On some continuous time algorithms in optimization and game theory

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To the memory of Bill Sandholm

Optimum and equilibria

The general framework is as follows:
V is a normed vector space, finite dimensional, with dual V^* and duality map $\langle V^*|V\rangle$,
X is a compact convex subset of V.

Convex optimization

Given $f : V \to I\!R$ convex and $\mathscr{C}^1$, the elements $\hat{x}$ realizing :

$$\min_X f(x)$$

are the solutions of:

$$\langle \nabla f(\hat{x}) | \hat{x} - y \rangle \leq 0, \qquad \forall y \in X. \tag{1}$$

Variational inequalities

More generally if g is a continuous vector field from V to V^* (that will play the rôle of $-\nabla f$) we introduce iS (i is for internal), as the set of solutions, $\hat{x} \in X$, of the variational inequality:

$$\langle g(\hat{x}) | \hat{x} - y \rangle \geq 0, \qquad \forall y \in X.$$

$$\langle g(\hat{x}) | \hat{x} - y \rangle \geq 0, \qquad \forall y \in X. \tag{2}$$

Note that in an Hilbertian framework, the solutions of (2) are also the solutions of:

$$\Pi_X(\hat{x} + g(\hat{x})) = \hat{x} \tag{3}$$

where Π_C denotes the projection operator on a closed convex set C, or the solutions of:

$$\Pi_{TX(\hat{x})}(g(\hat{x})) = 0 \tag{4}$$

where $TC(x)$ is the tangent cône to C at $x \in C$.
Recall that:

$$\Pi_{TX(x)}(y) = \lim_{h \to 0} \frac{\Pi_X(x + h\, y) - x}{h}.$$

Product case

In the product case, with I finite:
$V = \prod_{i \in I} V^i, V^* = \prod_{i \in I} V^{i*}$,
$g^i : X = \prod_{i \in I} X^i \to V^{i*}, i \in I$

$$\langle g(x) | y \rangle = \sum_i \langle g^i(x) | y^i \rangle$$

and $\hat{x} \in iS$ iff:

$$\langle g^i(\hat{x}) | \hat{x}^i - y^i \rangle \geq 0, \qquad \forall y^i \in X^i, \forall i \in I.$$

Equilibria in games

Three basic classes of games where equilibria are solutions of variational inequalities are as follows, see e.g. Sorin and Wang (2016) [70]:

A) *Finite games*

I is the finite set of players.

S^i is the finite set of actions of player i and $X^i = \Delta(S^i),\ i \in I$.

Player i's payoff G^i is a map from $S = \Pi_j S^j \to I\!R$, extended by multilinearity to $X = \prod_j X^j$. VG^i denotes the associated vector payoff function so that: $G^i(x) = \langle x^i, VG^i(x^{-i})\rangle$.

An *equilibrium*, Nash (1950) [48], $x \in X$ is given by:

$$G^i(x) \geq G^i(y^i, x^{-i}), \quad \forall y^i \in X^i, \forall i \in I \tag{5}$$

thus is a solution of :

$$\langle VG(x), x - y\rangle = \sum_{i \in I} \langle VG^i(x^{-i}), x^i - y^i\rangle \geq 0, \quad \forall y \in X. \tag{6}$$

This corresponds to iS for the vector field $g(x) = \{g^i(x) = VG^i(x^{-i}), i \in I\}$.

B) *Concave* $\mathscr{C}^1$ *games*

I is the finite set of players with action sets $\{X^i\}$ and payoff functions $\{H^i\}$, $i \in I$.

Assume: X^i is convex compact and $H^i : X = \prod_{j\in} X^j \to$ *IR* is of class $\mathscr{C}^1$ and concave with respect to x^i, $\forall i \in I$.

An *equilibrium*, Nash (1951) [49], is a profile $x \in X$ satisfying:

$$H^i(x) \geq H^i(y^i, x^{-i}), \quad \forall y^i \in X^i, \forall i \in I \tag{7}$$

which under our hypotheses is equivalent to:

$$\langle \nabla^i H^i(x), x^i - y^i \rangle \geq 0, \quad \forall y^i \in X^i, \forall i \in I \tag{8}$$

where ∇^i stands the gradient w.r.t. x^i.

In this framework the vector field is given by :

$g(x) = \{g^i(x) = \nabla^i H^i(x), i \in I\}$.

C) *Population games*

Consider a non atomic population with a finite set S of types.
A configuration is a probability vector $x \in X = \Delta(S)$ corresponding to the proportion of each type in the population.
The outcome is defined by a continuous function $\phi : S \times X \to I\!R$, thus where $\phi(p,x)$ the outcome of a member of the population, with type p, given the configuration x.
An *equilibrium*, Wardrop (1952)[76], is a profile $x \in X$ satisfying:

$$x^p > 0 \Rightarrow \phi(p,x) \geq \phi(q,x), \quad \forall p,q \in S \tag{9}$$

so that if p is present in the population at x, it is a best type.
An equivalent characterization of (9) is through the solutions of the variational inequality:

$$\sum_p \phi(p,x)(x^p - y^p) = \langle \phi(.,x), x-y \rangle \geq 0, \quad \forall y \in X$$

so that the corresponding vector field is $g(x) = \phi(.,x)$.
The extension to a finite set I of populations with for each population a finite set of types S^i is standard.

Recall that this representation of equilibria via variational inequalities is usual in transportation and congestion models, e.g. Dafermos (1980) [19], Dupuis and Nagurney (1993) [20], Smith (1979) [71] and in evolutionary game theory, e.g. Sandholm (2011) [62].

Dissipative property

Introduce the set eS (e for external) of solutions $\hat{x} \in X$ of

$$\langle g(y) | \hat{x} - y \rangle \geq 0, \qquad \forall y \in X. \tag{10}$$

Compare to :

$$\langle g(\hat{x}) | \hat{x} - y \rangle \geq 0, \qquad \forall y \in X.$$

Observe that eS is convex.
Moreover:
i) If g is continuous, then:

$$eS \subset iS.$$

ii) If g is dissipative ($-g$ is monotone) namely:

$$\langle g(x) - g(y) | x - y \rangle \leq 0, \qquad \forall x, y \in X$$

then:

$$iS \subset eS.$$

See Kinderlehrer and Stampacchia (1980) [34], Facchinei and Pang (2007) [21].

In general eS can be empty.

The proof of the non emptiness of iS is equivalent to the fixed point theorem.

On the other hand, if g is dissipative the proof that eS is non-empty follows from the (finite) min-max theorem, Minty (1967) [42].

Comparison: optimization/games

1. To correspond to a game $\Gamma(g)$, an algorithm for a vector field g on a product space $X = \prod_i X^i$ has to be decentralized: i.e. the dynamics for the component $x^i \in X^i$ is only function of the values of $g(x)$ on V^{i*}, namely g^i, i.e. it is uncoupled in the sense of Hart and Mas-Colell (2003) [26], hence of the form:

$$\dot{x}_t^i = T(x_t^i, g^i(x_t)), \forall i \in I.$$

This is the way the impact of the other players on player i is modelized and this also corresponds to his information.
Extension 1: less information = bandit framework with statistical tools to handle the information.
Extension 2: more information leads to "coordination" (common noise in the framework of MFG).

2. In convex optimization one considers both criteria:
- convergence of $f(x_t)$ to $\min_X f$
- convergence of the trajectory $\{x_t\}$ to $eS = iS = S = argmin_X f$
with 2 levels:
$d(x_t, S) \to 0$ or
x_t converges to $x^* \in S$.

In game theory one considers usually only trajectories.
An alternative quantitative criteria could be defined, in the spirit of the Nikaido (1955) function [53], as follows:

$$E_g(x) = \sup_{y \in X} \sum_i \langle g^i(x) | y^i - x^i \rangle$$

hence

$$iS = \{x \in X : E_g(x) = 0\}.$$

For zero-sum games this criteria is related to the duality gap.

3. Recall that the minimization of a $\mathscr{C}^1$ convex function f on X corresponds to a variational inequality with $g = -\nabla f$.
This implies two properties:
g is dissipative,
g is a gradient.
The analogous properties for games define two classes:
dissipative games,
potential games.

A game $\Gamma(g)$ is dissipative if g is dissipative.
This is related to the monotonicity requirement in Rosen (1965) [60].
The terminology is "stable" in Hofbauer and Sandholm (2009) [29] and "contractive" in Sandholm (2015) [63].

Fundamental example
In particular if $F : X = X^1 \times X^2 \to I\!R$ is $\mathscr{C}^1$ and concave/convex, the vector field $g = (\nabla^1 F, -\nabla^2 F)$ is dissipative, Rockafellar (1970) [59].
The elements of $iS = eS = S$ are optimal strategies of the associated 0-sum game.

We now define a potential for a vector field, see e.g. Sorin and Wang (2016) [70].

Definition 1

A real function W of class $\mathscr{C}^1$ on X, is a *potential* for g if there exist strictly positive functions μ^i on X, $i \in I$, such that:

$$\left\langle \nabla^i W(x) - \mu^i(x) g^i(x), y^i - x^i \right\rangle = 0, \quad \forall x \in X, \forall y^i \in X^i, \forall i \in I, \quad (11)$$

The game $\Gamma(g)$ corresponding to such g is a potential game.

Alternative previous definitions include:
Monderer and Shapley [43] for finite games,
Sandholm [61] for population games.

The following result is classical, see e.g. Sandholm (2011) [62].

Proposition 1.1

Let $\Gamma(g)$ be a game with potential W.
1. Every local maximum of W is an equilibrium of $\Gamma(g)$.
2. If W is concave on X, then any equilibrium of $\Gamma(g)$ is a global maximum of W on X.

Proof:
Since a local maximum x of W on the convex set X satisfies:

$$\langle \nabla W(x), x - y \rangle \geq 0, \quad \forall y \in X, \tag{12}$$

it follows from (11) that $\langle \mu^i(x) g^i(x), x^i - y^i \rangle \geq 0$ for all i and all $y \in X$. On the other hand, if W is concave on X, a solution x of (12) is a global maximum of W on X. ∎

Positive correlation

Given a dynamics, f decreases on trajectories if:

$$\frac{d}{dt} f(x_t) = \langle \nabla f(x_t) | \dot{x}_t \rangle \leq 0.$$

The analogous property for a vector field g is:

$$\langle g(x_t) | \dot{x}_t \rangle \geq 0.$$

In the framework of games, a similar condition was described in discrete time as Myopic Adjustment Dynamics, Swinkels (1993) [72] : if $x^i_{n+1} \neq x^i_n$ then $H^i(x^i_{n+1}, x^{-i}_n) > H^i(x^i_n, x^{-i}_n)$.

The corresponding property in continuous time corresponds to positive correlation, Sandholm (2011) [62]:

$$\dot{x}^i_t \neq 0 \Longrightarrow \langle g^i(x_t), \dot{x}^i_t \rangle > 0.$$

The use of this notion in potential games is a follows:

Proposition 1.2

Consider a game $\Gamma(g)$ with potential function W.
If the dynamics satisfies positive correlation, then W is a strict Lyapunov function.
All ω-limit points are rest points.

Proof:

Let $V_t = W(x_t)$ for $t \geq 0$. Then:

$$\dot{V}_t = \langle \nabla W(x_t) | \dot{x}_t \rangle = \sum_{i \in I} \langle \nabla^i W(x_t) | \dot{x}_t^i \rangle = \sum_{i \in I} \mu^i(x) \langle g^i(x_t) | \dot{x}_t^i \rangle \geq 0.$$

Moreover, $\langle g^i(x_t) | \dot{x}_t^i \rangle = 0$ holds for all i if and only if $\dot{x}_t = 0$.
One concludes by using Lyapunov's theorem (e.g. [31, Theorem 2.6.1]). ■

This result is proved by Sandholm (2001) [61] for his version of potential population game, see extensions in Benaim, Hofbauer and Sorin (2005) [9].
A similar property for fictitious play in discrete time is established in Monderer and Shapley (1996) [43].

We will show that this property holds for all the dynamics defined below.

No-regret dynamics

The next three dynamics satisfy the no-regret criteria defined, in general, as follows.
One associates to a process in the dual space $\{u_t \in V^*, t \geq 0\}$, a procedure in the primal $\{x_t \in X, t \geq 0\}$, where x_t is function of the past $\{(x_s, u_s), 0 \leq s < t\}$. The adequation of $\{x_t\}$ to $\{u_t\}$ is measured by a *regret function* defined by:

$$R_t(x) = \int_0^t \langle u_s | x - x_s \rangle ds, \qquad t \geq 0$$

and one will deal with procedures satisfying the "no-regret" property :

$$\sup_{x \in X} R_t(x) \leq o(t). \tag{13}$$

The algorithm is defined for a general bounded process $\{u_t\} \in V^*$, and we study its performance for two closed forms :

(I) equilibria or variational inequalities : where $u_t = g(x_t)$ for a continuous vector field $g : X \to V^*$,

(II) convex optimization : $u_t = -\nabla f(x_t), f$ convex $\mathscr{C}^1$ on X.

Basic properties

Assume that the procedure satisfies the no-regret property (13):

$$\sup_{x \in X} R_t(x) \leq o(t).$$

Lemma 2

If g is continuous and $x_s \to x$ then $x \in iS$.

Proof:

$$\frac{R_t(y)}{t} = \frac{1}{t}\int_0^t \langle g(x_s)|y - x_s\rangle ds \to \langle g(x)|y - x\rangle, \qquad \forall y \in X. \qquad (14)$$

∎

In particular if x is a stationary point for the dynamics, then $x \in iS$.

Define the time average trajectory:

$$\bar{x}_t = \frac{1}{t}\int_0^t x_s ds.$$

Lemma 3

If g is dissipative the accumulation points of $\{\bar{x}_t\}$ are in eS.

Proof:

$$\frac{R_t(y)}{t} = \frac{1}{t}\int_0^t \langle g(x_s)|y - x_s\rangle ds \geq \frac{1}{t}\int_0^t \langle g(y)|y - x_s\rangle ds = \langle g(y)|y - \bar{x}_t\rangle.$$

∎

Note that this shows that the existence of no-regret dynamics implies that eS is non empty for dissipative g.

Convex optimization

$S = iS = eS = argmin_X f$.

Use that:

$$\langle \nabla f(x_t) | y - x_t \rangle \leq f(y) - f(x_t)$$

to get with $u_t = -\nabla f(x_t)$:

$$\int_0^t [f(x_s) - f(y)] ds \leq \int_0^t \langle -\nabla f(x_s) | y - x_s \rangle ds = R_t(y)$$

which implies by Jensen's inequality:

$$f(\bar{x}_t) - f(y) \leq \frac{1}{t} \int_0^t [f(x_s) - f(y)] ds \leq \frac{R_t(y)}{t}. \qquad (15)$$

Lemma 4

i) The accumulation points of $\{\bar{x}_t\}$ belong to S.
ii) If $f(x_t)$ is decreasing, the accumulation points of $\{x_t\}$ or $\{x_n\}$ belong to S.

Basic tool

Definition 5

$P : I\!R^+ \times X \to I\!R^+$ is a *level function* if it satisfies:

$$\langle u_t | x_t - y \rangle \geq \frac{d}{dt} P(t; y).$$

Lemma 6

If there exists a level function, $R_t(y)/t$ *converges to* 0 *at a rate* $1/t$*. In particular the "no-regret" property holds.*

Proof:

$$R_t(y) = \int_0^t \langle u_s | y - x_s \rangle ds \leq P(0; y) - P(t; y).$$

∎

Lemma 7

Assume $\hat{y} \in eS$*, then* $P(t; \hat{y})$ *is decreasing:*

Proof:

$$\frac{d}{dt} P(t; \hat{y})) \leq \langle g(x_t) | x_t - \hat{y} \rangle \leq 0.$$

∎

1. Projection dynamics: Euclidean framework

Euclidean scalar product: $\langle , \rangle$.

Recall the *projected gradient descent*, Polyak (1987) [57],

$$x_{m+1} = argmax_X[\langle u_m, x\rangle - \frac{1}{2\eta_m}\|x - x_m\|^2] \tag{16}$$

with $u_m = -\nabla f(x_m)$, and step size $\eta_m > 0$ decreasing, or:

$$\begin{aligned} x_{m+1} &= argmin_X[\langle -u_m, x\rangle + \frac{1}{2\eta_m}\|x - x_m\|^2] \\ &= argmin_X\|x - (x_m + \eta_m u_m)\|^2 \end{aligned} \tag{17}$$

which corresponds to:

$$x_{m+1} = \Pi_X[x_m + \eta_m u_m], \tag{18}$$

thus with variational characterization:

$$\langle x_m + \eta_m u_m - x_{m+1}, y - x_{m+1}\rangle \leq 0, \forall y \in X. \tag{19}$$

The continuous time analog is given by the local projection dynamics, Dupuis and Nagurney (1993) [20], Lahkar and Sandholm (2008) [36]:

$$\langle u_t - \dot{x}_t, y - x_t \rangle \leq 0, \quad \forall y \in X \tag{20}$$

which is also:

$$\dot{x}_t = \Pi_{T_X(x_t)}(u_t). \tag{21}$$

since $T_X(x_t)$ is a cône.

Let:

$$V(t;y) = \frac{1}{2}\|x_t - y\|^2, \quad y \in X. \tag{22}$$

Proposition 2.1

V is a level function.

Proof:

One has:

$$\frac{d}{dt}V(t;y) = \langle \dot{x}_t, x_t - y \rangle \leq \langle u_t, x_t - y \rangle$$

by (20). ∎

Proposition 2.2

Assume g dissipative.
Then $\{\bar{x}_t\}$ converges to a point in eS.

Proof:
- The limit points of $\{\bar{x}_t\}$ are in eS by Lemma 3.
- $\|x_t - \hat{y}\|$ converges when $\hat{y} \in eS$ by Lemma 7.

Hence by Opial's lemma (1967) [54],
(In an Hilbert space, if $\|x_t - y\|$ converges for any y weak accumulation point of $\{x_t\}$ resp.$\{\bar{x}_t\}$, then x_t (resp. $\bar{x}_t$) weakly converges.)
$\bar{x}_t$ converges to a point in eS. ∎

Lemma 8

Positive correlation holds.

Proof:

$$\langle g(x_t), \dot{x}_t \rangle = \|\dot{x}_t\|^2$$

since $\langle u_t - \dot{x}_t, \dot{x}_t \rangle = 0$ by (21) and Moreau's decomposition, Moreau (1965) [45]. ∎

Convex case

Lemma 9

i) $\{x_t\}$ *converges to a point in* S*.*
ii) $f(x_t)$ *decreases to* $\min_X f$ *with speed* $O(1/t)$*.*

Proof:
i) Lemma 4 and Lemma 8 imply that the accumulation points of $f(x_t)$ are in S. Then using Lemma 7, Opial's Lemma applies.
ii) Follows from Lemma 6. ■

Remark

All the results of this section extend to the Hilbert case, convergence being understood as weak convergence.

2. Mirror descent : differential/incremental approach

The assumptions are:
$H : V \to I\!R$, strictly convex, $\mathscr{C}^2$
X, compact and convex, $\subset domH$.
The Bregman distance associated to H is:

$$D_H(x,y) = H(x) - H(y) - \langle \nabla H(y) | x - y \rangle \, (\geq 0). \quad (23)$$

The discrete version is the *mirror descent algorithm*, Nemirovski and Yudin (1983) [50], Beck and Teboulle (2003) [8] defined, for $u_m = \nabla f(x_m)$, by:

$$x_{m+1} = argmax_X \{ \langle u_m | x \rangle - (1/\eta_m) D_H(x, x_m) \}. \quad (24)$$

which gives:

$$\langle \nabla H(x_m) + \eta_m u_m - \nabla H(x_{m+1}) | x - x_{m+1} \rangle \leq 0, \forall x \in X. \quad (25)$$

The continuous time procedure, mirror descent, satisfies:
$x_t \in X$ and:

$$\langle u_t - \frac{d}{dt}\nabla H(x_t) | x - x_t \rangle \leq 0, \forall x \in X. \tag{26}$$

The analysis of the previous Section corresponds to the case:

$$H(x) = \frac{1}{2}\|x\|^2.$$

Proposition 2.3

$P(t;y) = D_H(y, x_t)$ *is a level function.*

Proof:

Note the following relation:

$$\frac{d}{dt}D_H(y, x_t) = -\langle \frac{d}{dt}\nabla H(x_t) | y - x_t \rangle \tag{27}$$

so that (26) implies

$$\frac{d}{dt}D_H(y, x_t) \leq \langle u_t | x_t - y \rangle. \tag{28}$$

Interior trajectory

The use of a specfic function H adapted to X, with $\|\nabla H(x)\| \to +\infty$ as $x \to \partial X$ allows to produce a trajectory that leaves $int\, X$ invariant and (26) leads to an equality:

$$\frac{d}{dt}\nabla H(x_t) = u_t \tag{29}$$

thus:

$$\nabla H(x_t) = \int_0^t u_s ds \tag{30}$$

and then:

$$\dot{x}_t = \nabla^2 H(x_t)^{-1} u_t. \tag{31}$$

$\nabla^2 H(x)$ induces a Hessian Riemannian metric as analyzed in Alvarez, Bolte and Brahic (2004) [1] and in Mertikopoulos and Sandholm (2018) [40] for games.

Lemma 10

Positive correlation holds.

Proof :

$$\langle g(x_t) | \dot{x}_t \rangle = \langle g(x_t) | \nabla^2 H(x_t)^{-1} g(x_t) \rangle \geq 0.$$

■

To prove convergence in the optimization case, one uses the following properties:
[H1] if $z^k \to y^* \in S$ then $D_H(y^*, z^k) \to 0$.
For example H L-smooth and then:

$$0 \leq D_H(x,y) \leq \frac{L}{2}\|x-y\|^2$$

[H2] if $D_H(y^*, z^k) \to 0, y^* \in S$ then $z^k \to y^*$.
For example H β-strongly convex and then:

$$D_H(x,y) \geq \frac{\beta}{2}\|x-y\|^2$$

Proposition 2.4

If H is smooth and strongly convex, $\{x_t\}$ converges to some $x^* \in S$.

Proof:
Consider an accumulation point x^* of $\{x_t\}$. Then $x^* \in S$ and thus $D_H(x^*, x_t)$ is decreasing by Lemma 7. Since this sequence is decreasing to 0 on a subsequence $x_{t_k} \to x^*$ by [H1], it is decreasing to 0 hence by [H2] $x_t \to x^*$. ∎

In the framework of games the function

$$h(x) = \sum_{p \in S} x^p Log x^p$$

defined on the simplex $X = \Delta(S)$ leads to the *replicator dynamics* on $intX$, Taylor and Jonker (1978) [73], Hofbauer and Sigmund (1998) [31], Sorin (2009) [66], Sorin (2020) [68]. The corresponding Riemannian metric is introduced in Shahshahani (1979) [65].

Recall that $h(x) = \frac{1}{2}\|x^2\|$ leads to the *local/direct projection dynamics*, for a comparison see Sandholm, Dokumaci and Lahkar (2008) [64].

3. Dual averaging: integral/cumulative approach

This corresponds to the continuous version of dual averaging, Nesterov (2009) [52].
We follow the analysis in Kwon and Mertikopoulos (2017) [35].
Assumptions:
$h : V \to \mathbb{R} \cup \{+\infty\}$ bounded strictly convex s.c.i. with $dom\, h = X$
Introduce :

$$U_t = \int_0^t u_s ds.$$

Define x_t as the argmax (on V or X) of:

$$\langle U_t | x \rangle - h(x).$$

Let $h^*(w) = \sup_{x \in V}[\langle w | x \rangle - h(x)]$ be the Fenchel conjugate of h.
h^* is differentiable.
The dynamics can be written as:

$$x_t = \nabla h^*(U_t) \in X \qquad (32)$$

Define, for $y \in X$:

$$W(t;y) = h^*(U_t) - \langle U_t | y \rangle + h(y). \tag{33}$$

Proposition 2.5

$W(t;y)$ is a level function.

Proof:

$W(t;y) \geq 0$ by Fenchel inequality.

Note that:

$$\frac{d}{dt} h^*(U_t) = \langle u_t | \nabla h^*(U_t) \rangle = \langle u_t | x_t \rangle \tag{34}$$

by (32) thus:

$$\frac{d}{dt} W(t;y) = \langle u_t | x_t - y \rangle$$

■

Lemma 11

Positive correlation holds.

Proof:

$$\langle g(x_t) | \dot{x}_t \rangle = \langle g(x_t) | \nabla^2 h^*(U_t)(u_t) \rangle$$

with $u_t = g(x_t)$. ∎

In the interior smooth case both level functions of the last two sections are the same, since:

$$x_t = \nabla h^*(U_t), \qquad \nabla h(x_t) = U_t, \qquad h^*(U_t) + h(x_t) = \langle U_t | x_t \rangle$$

and

$$\begin{aligned} D_h(y, x_t) &= h(y) - h(x_t) - \langle \nabla h(x_t) | y - x_t \rangle \\ &= h(y) + h^*(U_t) - \langle U_t | x_t \rangle - \langle \nabla h(x_t) | y - x_t \rangle \\ &= h(y) + h^*(U_t) - \langle U_t | y \rangle \\ &= W(t; y) \end{aligned}$$

For more properties see Mertikopoulos and Sandholm (2016) [39].

4. Conditional gradient and global projection

The same dynamics appears as conditional gradient in convex optimization : Antipin (1994) [2], Bolte (2003)[13]) and as global/target projection dynamics in operations research and game theory : Friesz et al. (1994) [23], Tsakas and Voorneveld (2009) [74].
The dynamics is given in an Euclidean space by:

$$\dot{x}^i = \Pi_{X^i}[x^i + g^i(x)] - x^i, \quad i \in I$$

which comes from the discrete process:

$$x^i_{n+1} - x^i_n = \lambda_n [\Pi_{X^i}[x^i_n + g^i(x_n)] - x^i_n].$$

The rest points are the set iS.

The variational expression is :

$$\langle g^i(x_t) - \dot{x}_t^i, z^i - (\dot{x}_t^i + x_t^i) \rangle \leq 0, \qquad \forall z^i \in X^i. \tag{35}$$

Lemma 12

Positive correlation holds.

Proof:
Use (35) for $z_t^i = x_t^i$:

$$\langle g^i(x_t) - \dot{x}_t^i, -\dot{x}_t^i) \rangle \leq 0$$

hence :

$$\langle g^i(x_t), \dot{x}_t^i) \rangle \geq \|\dot{x}_t^i\|^2. \tag{36}$$

■

Optimization

Consider the case of a convex $\mathscr{C}^1$ function f.
We follow Antipin and Bolte.

Proposition 3.1

i) $f(x_t)$ *decreases to* $\min_X f$ *with speed* $\frac{1}{t}$.
ii) $\{x_t\}$ *converges to a point in* S.

Proof :
i) From convexity :

$$f(z) - f(x_t) \geq \langle \nabla f(x_t), z - x_t \rangle$$

one obtains :

$$f(x_t) - f(z) \leq \langle \nabla f(x_t), -\dot{x}_t \rangle + \langle \nabla f(x_t), \dot{x}_t + x_t - z \rangle$$

thus using (35) one deduces:

$$f(x_t) - f(z) \leq \langle \nabla f(x_t), -\dot{x}_t \rangle - \langle \dot{x}_t, \dot{x}_t + x_t - z \rangle.$$

This gives:

$$\frac{d}{dt}[\frac{1}{2}\|x_t - z\|^2 + f(x_t)] \leq -\frac{1}{2}\|\dot{x}_t\|^2 + f(z) - f(x_t)$$

hence for $\hat{z} \in S$, $t \mapsto \frac{1}{2}\|x_t - \hat{z}\|^2 + f(x_t)$ is decreasing.
Now invoking (36) which implies $t \mapsto f(x_t)$ decreasing, $\|x_t - \hat{z}\|^2$ converges.
Integrating and using again $t \mapsto f(x_t)$ decreasing, implies:

$$\frac{1}{2}\|x_t - z\|^2 + f(x_t) + t[f(x_t - f(z)] \leq \frac{1}{2}\|x_0 - z\|^2 + f(x_0)$$

hence the convergence of $f(x_t)$ to $\min_X f$ with speed $\frac{1}{t}$.
ii) Let now z^* be an accumulation point of $\{x_t\}$. Thus $z^* \in S$ and $\|x_t - z^*\|$ converges, thus by Opial's lemma $\{x_t\}$ converges. ∎

Vector field

Consider now a smooth dissipative vector field g.
Let $M(x,y) = \frac{1}{2}\|(x+g(x))-y)\|^2$, $L(x,y) = M(x,x) - M(x,y)$ and $H(x) = \sup_{y \in X} L(x,y)$.

Proposition 3.2

H is a Lyapounov function for S.

Proof :
Note that $H(x) = M(x,x) - M(x,\hat{y}(x))$ with $\hat{y}(x) = \Pi_X(x+g(x))$.
Moreover $H \geq 0$ and by definition of the projection Π_X, equality holds if and only if $x = \hat{y}(x) = \Pi_X(x+g(x))$.

By the Enveloppe theorem:

$$\begin{aligned}\frac{d}{dt}H(x_t) &= \langle \nabla H(x_t), \dot{x}_t \rangle \\ &= \langle \nabla_x L(x_t, \hat{y}(x_t)), \dot{x}_t \rangle \\ &= \langle -g(x_t) + (\hat{y}(x_t) - x_t), \hat{y}(x_t) - x_t \rangle + (\hat{y}(x_t) - x_t) J_g(x_t)(\hat{y}(x_t) - x_t) \\ &\leq 0,\end{aligned}$$

where J_g is the Jacobian of g.
The first term is negative since $\hat{y}(x) = \Pi_X(x + g(x))$. The second term is negative because g is dissipative.
Thus H is a Lyapunov function.
Note that H is a strict Lyapunov function when g is strictly dissipative. ■
This result is proved by Pappalardo and Passacantando (2004) [56] in one-population game setting.

All the properties of this section extend to the Hilbert framework, convergence being understood as weak convergence.

5. Frank-Wolfe and best reply

The best reply dynamics is usually defined trough the best reply correspondence.
In the framework of a strategic game with payoff function $H^i : X = \prod_j X^J \to I\!R$, the definition is :

$$br^i(x) = \{y^i \in X^i;\, H^i(y^i, x^{-i}) \geq H^i(z^i, x^{-i}), \forall z^i \in X^i\}.$$

Note that it is independent of x^i.
In our framework we will use the first order optimality condition (which corresponds to a local linearization of the payoff) so that:

$$BR^i(x) = \{y^i \in X^i; \langle g^i(x) | y^i - z^i \rangle \geq 0, \forall z^i \in X^i\}$$

where g^i is the vector field involved to define an equilibrium.
(Remark that in the case of a finite game - with multilinear extension - both definitions agree).

The best reply dynamics, Gilboa and Matsui (1991) [24], is defined by the differential inclusion:

$$\dot{x}^i \in BR^i(x) - x^i, \quad i \in I,$$

thus: $\dot{x} = y(x) - x$ with $y(x) \in BR(x)$ and the rest points are iS.

Note that if $g = -\nabla f$, this corresponds precisely to the Frank-Wolfe algorithm, Frank and Wolfe (1956) [22].
Recall that the discrete time version is:

$$x_{n+1} - x_n = \lambda_n [y(x_n) - x_n]$$

where $y(x) \in argmin\{\langle \nabla f(x) | z \rangle; z \in X\}$.
Hence the continuous time version, corresponding to $\lambda_n = O(\frac{1}{n})$ takes the form:

$$\dot{x}_t = \frac{1}{t}[y(x_t) - x_t]$$

which gives the previous equation by a change of time.

Lemma 13

Positive correlation holds.

Proof:

$$\langle g^i(x_t), \dot{x}_t^i) \rangle = \langle g^i(x_t), y^i(x_t) - x_t^i \rangle \geq 0$$

■

Proposition 4.1

$f(x_t)$ decreases to $\min_X f$.

Proof:
Letting $\delta_t = f(x_t) - f(\hat{x})$ with $\hat{x} \in S$ one has $\delta_t \geq 0$ and:

$$\begin{aligned}
\dot{\delta}_t &= \frac{d}{dt} f(x_t) \\
&= \langle \nabla f(x_t) | y(x_t) - x_t \rangle \\
&\leq \langle \nabla f(x_t) | \hat{x} - x_t \rangle \\
&\leq f(\hat{x}) - f(x_t) \\
&= -\delta_t
\end{aligned}$$

hence $\delta_t \leq \delta_0 e^{-t}$, thus speed of convergence of the order $O(\frac{1}{t})$ before the time change. ∎

Vector field

Let $Q(x) = \sup_{z \in X} M(x,z)$ with $M(x,z) = \langle g(x) | z - x \rangle$, for $x, z \in X$.
Thus $Q \geq 0$ and $Q^{-1}(0) = iS$.

Proposition 4.2

Assume g dissipative.
Then Q is a strict Lyapounov function for S.

Proof:
$Q(x) = M(x, y(x))$ with $y(x) \in BR(x)$.
By the Enveloppe Theorem:

$$\begin{aligned} \langle \nabla Q(x) | \dot{x} \rangle &= \langle \nabla_x M(x, y(x)) | \dot{x} \rangle \\ &= \langle -g(x) + (y(x) - x) J_g(x), y(x) - x \rangle \\ &= -Q(x) + [y(x) - x] J_g(x) [y(x) - x] \\ &\leq 0. \end{aligned}$$

The second term is negative because g is dissipative. Then the equality holds if and only if $Q(x) = 0$. Therefore Q is a strict Lyapunov function. More precisely if $\alpha_t = Q(x_t)$, $\dot{\alpha}_t \leq -\alpha_t$ and $\alpha_t \leq \alpha_0 \, e^{-t}$.

This result appears for population games in Hofbauer and Sandholm (2009)[29]. Compare with Fictitious Play/Best Reply, for two-players zero-sum games and duality gap, Hofbauer and Sorin (2006) [32].

Final comments

A) There are strong analogies between dynamics in both areas.
Properties for convex optimization are a test for games (information, stochastic perturbation , ...)
Properties for games are a proof of robustness for optimization.

B) Properties for the average processes $\{\bar{x}_t\}$ in cases (1, 2, 3).
Recall Ficitious Play:

$$Y^i_{n+1} \in br^i(\bar{Y}_n)$$

$$\bar{Y}^i_{n1} - \bar{Y}^i_n \in \frac{1}{n+1}[br^i(\bar{Y}_n) - \bar{Y}^i_n]$$

so that x_t in the continuous time best reply dynamics corresponds to $\bar{Y}_n$ in discrete time.
Explicit link for two person finite games analyzed Hofbauer, Sorin and Viossat (2009) [33].

C) Extensions
1) Nash's conditions : H^i continuous quasi-concave, Hofbauer and Sorin (2006) [32], Barron, Goebel and Jensen (2010) [7] . Link with subgradient dynamics and maximal monotone operators, Brézis (1973) [15], Bruck (1975) [18].
2) Comparison in discrete time : step size, descent lemma, regularity, Sorin (2021) [69].
3) Stochastic framework : tools of stochastic approximation, Benaim, Hofbauer and Sorin (2005) [9], (2006) [10]

D) General comments

Positive results for:

- (pseudo) gradient dynamics
- dissipative case:

i) extension of the initial dynamics, Brown and von Neumann (1950) [17].

ii) convex set of equilibria

iii) similar to learning procedures (coarse or correlated equilibria)

No convergence in general:

Hart and Mas-Colell (2003) [26],

Hofbauer's example= 2 rooms, 3 players,

Viossat (2014)[75]

E) Extensions via 2 approaches:

Alternative notion of "equilibrium", associated classes of dynamics and games
ESS and replicator dynamics, Maynard Smith (1982) [38]
Hofbauer and Sigmund (1990) [30], (1998) [31]
same spirit: Mertikopoulos and Zhou (2019) [41].

Define "natural" dynamics and study the associated attractors: analysis of ICT sets, Benaim, Hofbauer and Sorin (2005) [9] (2006) [10], (2012) [11].
Stable component $\neq$ subset of rest points.

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