Charges and Bets: A General Characterisation of Common Priors

joint with Ziv Hellman

Miklós Pintér

Corvinus Center for Operations Research

Mathematical Game Theory Seminar 2020

The finite case (Samet, 1998)

Definition: A type space for a set of agents $I = \{1, \dots, n\}$ is a tuple $(\Omega, (\Pi_i, t_i)_{i \in I})$, where

- Ω is a finite set of states,
- Π_i is a partition of Ω, and
- t_i is a function $t_i \colon \Omega \to \Delta(\Omega)$, which associates with each state ω the type of i at ω, i.e., a point in Δ^Ω, the simplex in $\mathbb{R}^\Omega$, which we consider as the set of probability distributions over Ω. The type function t_i satisfies the following two conditions:
  - for each $\omega \in \Omega$, $t_i(\omega)(\Pi_i(\omega)) = 1$, where $\Pi_i(\omega)$ is the element of the partition Π_i which contains ω,
  - t_i is constant over each element of Π_i.

The finite case (Samet, 1998) I

Example: Consider the following type space $(\Omega, (\Pi_i, t_i)_{i \in I})$, where

- the state space is $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$,
- the players set is $I = \{1, 2\}$,
- the two partitions are $\Pi_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and $\Pi_2 = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$,
- the two type functions $t_1 \colon \Omega \to \Delta(\Omega)$ are such that

$$t_1(\omega_1) = t_1(\omega_2) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right), \tag{1}$$

and

$$t_1(\omega_3) = t_1(\omega_4) = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right). \tag{2}$$

Moreover, let $t_2 \colon \Omega \to \Delta(\Omega)$ be such that

The finite case (Samet, 1998) II

$$t_2(\omega_1) = t_2(\omega_3) = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right), \tag{3}$$

and

$$t_2(\omega_2) = t_2(\omega_4) = (0, 1, 0, 0)\,. \tag{4}$$

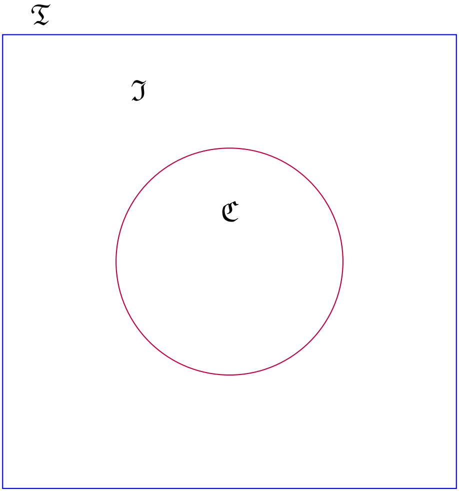
The finite case (Samet, 1998)

Definition: Take a type space $(\Omega, (\Pi_i, t_i)_{i \in I})$. Then the probability distribution $P_i \in \Delta(\Omega)$ is a prior for player i, if for each $\pi_i \in \Pi_i$ such that $P_i(\pi_i) > 0$ it holds that $t_i(\omega)(\cdot) = P_i(\cdot \mid \pi_i)$, $\omega \in \pi_i$. Let $\mathcal{P}_i$ denote the set of player i's priors.

A probability distribution P is a common prior, if $P \in \cap_{i \in I} \mathcal{P}_i$, in words, a common prior is a prior for every player.

The finite case (Samet, 1998)

Symbolically, we can partition the space of type spaces $\mathfrak{T}$ over Ω epistemically into consistent type spaces $\mathfrak{C}$ and inconsistent ones $\mathfrak{I}$



The finite case (Samet, 1998)

Example: Considering the type space above, it does not attain a common prior. Suppose for contradiction that P is a common prior, then by Equation (1)

$$P(\{\omega_1\}) = P(\{\omega_2\})$$

and by Equation (3)

$$P(\{\omega_1\}) = P(\{\omega_3\})$$

and by Equation (2)

$$P(\{\omega_3\}) = P(\{\omega_4\})$$

which contradicts Equation (4) $(P(\{\omega_2\}) \neq P(\{\omega_4\}))$.

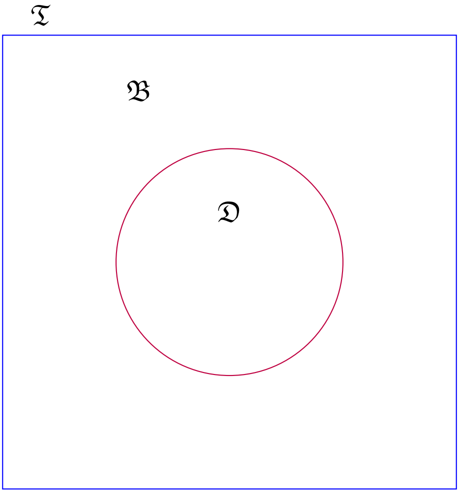
The finite case (Samet, 1998)

Definition: A collection of functions $(f_i)_{i \in I}$, $f_i \colon \Omega \to \mathbb{R}$, $i \in I$, is a bet for the type space $(\Omega, (\Pi_i, t_i)_{i \in I})$, if $\sum_{i \in I} f_i = 0$. The bet $(f_i)_{i \in I}$ is an agreeable bet if for each $i \in I$ and $\omega \in \Omega$ it holds that

$$\int f_i \ \mathrm{d}\, t_i(\omega) > 0.$$

The finite case (Samet, 1998)

We can partition $\mathfrak{T}$ over Ω behaviourally into type spaces that admit agreeable bets $\mathfrak{B}$ and ones that do not $\mathfrak{D}$



The finite case (Samet, 1998)

Example: Considering the type space above, let

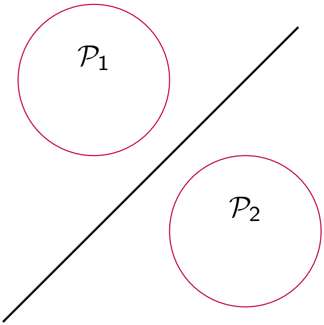
$$f_1(\omega) = \begin{cases} 2 & \text{if } \omega = \omega_1, \\ -1 & \text{if } \omega = \omega_2, \\ -3 & \text{if } \omega = \omega_3, \\ 4 & \text{if } \omega = \omega_4. \end{cases}$$

Moreover, let $f_2 = -f_1$. Then (f_1, f_2) is an agreeable bet.

The finite case (Samet, 1998)

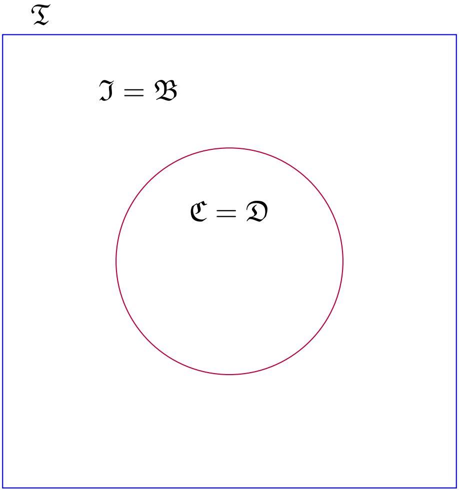
Theorem: A type space $(\Omega, (\Pi_i, t_i)_{i \in I})$ attains a common prior if and only if there does not exist an agreeable bet for it.

Let $\mathcal{P}_1, \mathcal{P}_2$ be respectively the (compact and convex) sets of priors of player 1 and player 2



The finite case (Samet, 1998)

The theorem above tell us that we get the same partition of the space of type spaces in both cases: $\mathfrak{C} = \mathfrak{D}$



The problem we address

- What can we say in the infinite setting?
- Can we generalize Samet (1998)'s characterization to the infinite setting?
- If yes, then how?

Motivation

- Aumann (1976)'s No Disagreement, or No Betting Theorem,
- No Betting Theorem (proved independently by several researchers in 1990s),
- This has had broad implications in the study of speculative trade, interactions between Bayesian agents, Bayesian persuasion, and many other fields,
- CPA (Common Prior Assumption) is central in economic theory, but how to test CPA?

Feinberg (2000)'s counterexample

Anne	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\dots$	
Ben	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\dots$

In this type space:

- There is no common prior,
- There is no agreeable bet either.

Hopes (positive results)

- By Feinberg (2000) and Heifetz (2006) if
  - The state space is compact,
  - The type functions are continuous,
  - The beliefs are (compact) regular probability measures,
  - The bets are continuous functions,

  then the epistemic and behavioral equivalence is restored: *A type space attains a common prior if and only if there does not exist an agreeable bet for it*.
- For countable type spaces, Lehrer and Samet (2014) present a three-levelled epistemic classification – weakly belief consistent, belief consistent, strongly belief consistent – equivalent to three behavioural properties.

What to generalize?

Questions to address:

- What should be the state space? Options: a topological space, a measurable space, etc.,
- What should be the beliefs? Options: probability measures, regular probability measures, probability charges, etc.,
- What should be the type functions? Options: continuous functions, measurable functions, some integrable functions, etc.,
- What should be the priors? Options: 'prior-first' or 'posterior-first',
- What should be the bets? Options: bounded continuous functions, bounded measurable functions, some integrable functions, etc.

Brief Review of Finite Additivity Concepts

- If $\mathcal{A}$ is a field on Ω then $(\Omega, \mathcal{A})$ is called a *chargeable space*,
- A set function $\mu \colon \mathcal{A} \to \mathbb{R}$ on a chargeable space is a *probability charge* if μ is non-negative, additive, and $\mu(\Omega) = 1$,
- If $\mathcal{A}$ is a σ-field then $(\Omega, \mathcal{A})$ is a *measurable space*,
- If probability charge μ on a measurable space is σ-additive it is a *probability measure*,
- Probability measures are special cases of probability charges.

Integral if the integrator is a charge

Take a charge space $(X, \mathcal{A}, \mu)$, and let $S(X, \mathcal{A})$ denote the class of step functions over $(X, \mathcal{A})$. For a bounded function $f\colon X \to \mathbb{R}$ define the lower integral of f by

$$I_*(f) = \sup\left\{\int s \,\mathrm{d}\,\mu\colon s \in S(X, \mathcal{A}) \text{ and } s \leq f\right\},$$

and the upper integral by

$$I^*(f) = \inf\left\{\int s \,\mathrm{d}\,\mu\colon s \in S(X, \mathcal{A}) \text{ and } s \geq f\right\}.$$

A function f is μ-integrable, if $I_*(f) = I^*(f)$.

A function f is $\mathcal{A}$-integrable, if it is integrable by any probability charge over $(X, \mathcal{A})$.
The dual of the $\mathcal{A}$-integrable functions is $\mathrm{ba}(X, \mathcal{A})$, and the set of $\mathrm{pba}(X, \mathcal{A})$ is weak* compact.

Our model

- We suppose throughout the existence of a a chargeable space $(\Omega, \mathcal{M})$.
- The elements of the field $\mathcal{M}$, the *general epistemic field*, serve as the *events* of interest in our model.
- The model also includes a set of players N (which is not necessarily assumed to be a finite set).
- Each player $i \in N$ is associated with a field $\mathcal{M}_i \subset \mathcal{M}$ (i's *private epistemic field*), which together with Ω forms a chargeable space $(\Omega, \mathcal{M}_i)$.

Our model

Definition: A *type function* is a mapping $t_i : \Omega \times \mathcal{M} \to [0, 1]$ satisfying

1. $t_i(\omega, \cdot)$ is a probability charge on $\mathcal{M}$ for all $\omega \in \Omega$,
2. $t_i(\cdot, E)$ is $\mathcal{M}_i$-integrable for each event E in the field $\mathcal{M}$.

Definition: A tuple $((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ is called a *type space* if for each player $i \in N$, player i's type function t_i satisfies the property that for each $E \in \mathcal{M}_i$, for each $\omega \in E$, $t_i(\omega, E) = 1$.

Our model

Definition: Let $T = ((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ be a type space, and let $i \in N$ be a player. Then a probability charge $P_i \in \mathrm{pba}(\Omega, \mathcal{M})$ is a *prior* of player i (relative to T) if for each $A \in \mathcal{M}$ and $B \in \mathcal{M}_i$

$$P_i(A \cap B) = \int_B t_i(\cdot, A) \, \mathrm{d} P_i.$$

Π_i will be used to denote the set of player i's priors (with T understood from context).

Definition: A charge $P \in \mathrm{pba}(\Omega, \mathcal{M})$ is a common prior, if

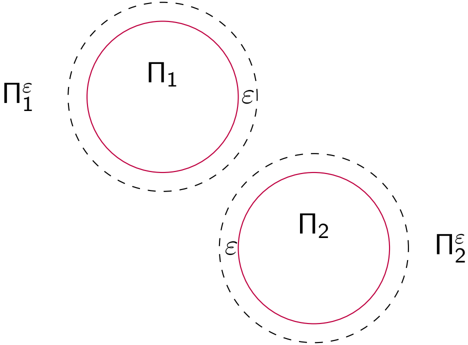
$$P \in \bigcap_{i \in N} \Pi_i.$$

Lemma: Let $((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ be a type space and let $i \in N$ be a player. Then Π_i, the set of player i's priors, is the weak* closure of the convex hull of i's types, that is,

$$\Pi_i = \overline{\mathrm{conv}(\{t_i(\omega, \cdot) \colon \omega \in \Omega\})}^{*}.$$

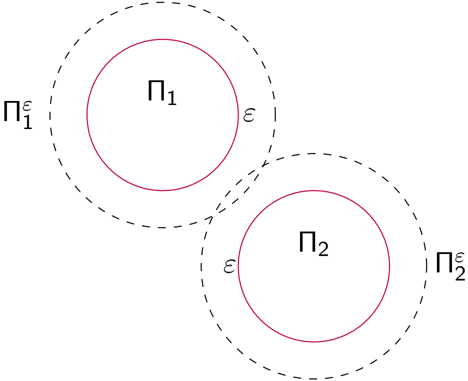
‘Thickening’

Imagine ‘ε thickening’ the sets of priors by adding more probability charges that are ‘ε distant’ (by the TV norm) from the boundaries of the sets. Denote the resulting sets Π_i^ε



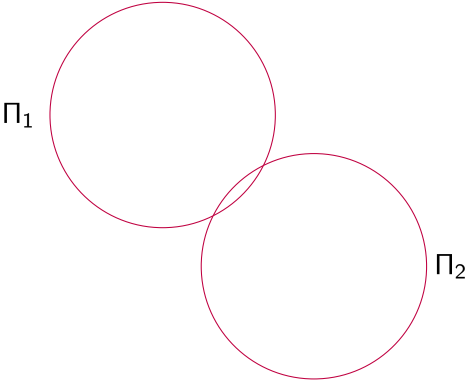
‘Thickening’

Can ask how much $\varepsilon > 0$ thickening until there is an intersection?
This intuitively measures ‘how far’ Π_1 and Π_2 are from each other



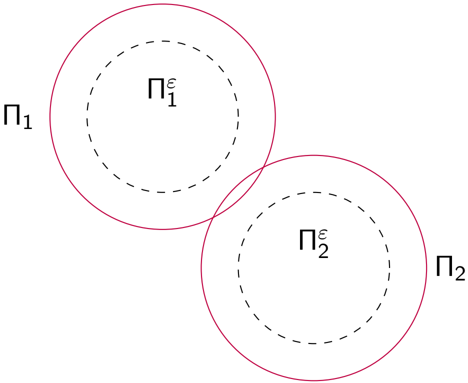
‘Thinning’

For $\varepsilon < 0$, imagine ‘thinning’ sets of priors by removing probability charges that are near the boundary



‘Thinning’

Can ask how much thinning until Π_1^ε and Π_2^ε are disjoint?



Our model

Definition: For each player i and each $\varepsilon \in [-1,1]$ define $\Pi_i^\varepsilon =$

$$\begin{cases} \overline{(\Pi_i + \mathcal{O}_{TV}(0,\varepsilon))}^* \cap \mathrm{pba}(\Omega,\mathcal{M}) & \text{if } \varepsilon > 0, \\ \Pi_i & \text{if } \varepsilon = 0, \\ \overline{\{\mu \in \mathrm{pba}(\Omega,\mathcal{M}) \colon \mathcal{O}_{TV}(\mu,-\varepsilon) \cap \mathrm{pba}(\Omega,\mathcal{M})) \subseteq \Pi_i\}}^* & \text{if } \varepsilon < 0, \end{cases}$$

A charge P is an ε-*common prior* for $\varepsilon \in [-1,1]$ if

$$P \in \cap_{i \in N} \Pi_i^\varepsilon.$$

Our model

<u>*Definition*</u>: Let $T = ((\Omega, \mathcal{M}), \{\Omega, \mathcal{M}_i\}_{i \in N}, (t_i)_{i \in N})$ be a type space. A set of $\mathcal{M}$-measurable functions $\overline{f} = \{f_{i_1}, \ldots, f_{i_n}\}$, for a finite index set $i_1, \ldots, i_n \in N$, is a *bet* if $\sum_{m=1}^{n} f_{i_m} = 0$.

A bet is an *agreeable bet* (relative to T) if there exists $\alpha \in \mathbb{R}$ such that

$$\int f_{i_m} \; \mathrm{d}\, t_{i_m}(\omega, \cdot) \geq \alpha > 0,$$

for every state $\omega \in \Omega$ and every player i_m with $m \in \{1, \ldots, n\}$.

Our model

Definition: Let $T = ((\Omega, \mathcal{M}), \{\Omega, \mathcal{M}_i\}_{i \in N}, (t_i)_{i \in N})$ be a type space. A set of functions $\overline{f} = \{f_{i_1}, \ldots, f_{i_n}\} \in B(\Omega, \mathcal{M})$, for a finite index set $i_1, \ldots, i_n \in N$, is a *bet* if $\sum_{m=1}^{n} f_{i_m} = 0$.
A bet is an ε*-agreeable bet* (relative to T) for $\varepsilon \in [-1, 1]$ if there exists $\alpha \in \mathbb{R}$ such that

$$\int f_{i_m} \; \mathrm{d}\, t_{i_m}(\omega, \cdot) \geq \alpha > \varepsilon \|f_{i_m}\|_{\sup},$$

for every state $\omega \in \Omega$ and every player i_m with $m \in \{1, \ldots, n\}$.

The main result

Theorem: Let $T = ((\Omega, \mathcal{M}), \{\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ be a type space and $\varepsilon \in [-1, 1]$. Then only one of the following two cases is possible:

- T admits an ε-common prior.
- There exists an ε-agreeable bet.

The proof (for $\varepsilon = 0$)

Theorem: Let $K_1, \ldots, K_n$ be non-empty, compact, convex sets in a locally convex topological vector space X such that $0 \notin K_m$, $m = 1, \ldots, n$. Then $\cap_{m=1}^{n} \operatorname{cone}(K_m) = \{0\}$, where $\operatorname{cone}(B) = \{\alpha x \colon \alpha \geq 0, \text{ and } x \in B\}$, if and only if there exist continuous linear functionals $f_1, \ldots, f_n$ over X, and $\alpha > 0$, such that $f_m(x) \geq \alpha$ for each $m = 1, \ldots, n$ and for all $x \in K_m$, and in addition $\sum f_m = 0$.

The proof (for $\varepsilon = 0$) I

If: Suppose by contradiction that $\cap_{m=1}^{n} \mathrm{cone}(K_m) \neq \{0\}$ and that at the same time there exist linear functionals $f_1, \ldots, f_n$ such that $f_m(x) \geq \alpha > 0$, for each $m = 1 \ldots, n$ and for each $x \in K_m$, with $\sum_{m=1}^{n} f_m = 0$. Since $\cap_{m=1}^{n} \mathrm{cone}(K_m) \neq \{0\}$, there is an $x \neq 0$ such that $x \in \cap_{m=1}^{n} \mathrm{cone}(K_m)$; such an x then satisfies the property that there exist $\beta_1, \ldots, \beta_n > 0$ such that $\beta_m x \in K_m$ for $m = 1, \ldots, n$, and $\sum f_m(x) \geq \alpha \sum_{m=1}^{n} \frac{1}{\beta_m}$. This contradicts $\sum f_m(x) = 0$.

Only if: Let $\hat{K} = \{x \in K_1^{n-1} \colon x_1 = \ldots = x_{n-1},\ x_1 \in K_1)$ (so that $\hat{K}$ is an $n-1$ copy of K_1), and let $\tilde{K} = \mathrm{cone}(K_2) \times \ldots \times \mathrm{cone}(K_n)$. It is clear that $\tilde{K}$ is weakly* closed and convex and that $\hat{K}$ is weakly* compact and convex. Suppose that $\cap_{m=1}^{n} \mathrm{cone}(K_m) = \{0\}$. Then it follows from the definitions that $\tilde{K} \cap \hat{K} = \emptyset$, which implies that there exists a continuous linear functional g, a real number β, and $\varepsilon > 0$ such that $g(x) \geq \beta + \varepsilon$ for all $x \in \tilde{K}$ and $g(x) \leq \beta - \varepsilon$ for all $x \in \hat{K}$.

The proof (for $\varepsilon = 0$) II

Since $\tilde{K}$ contains the origin, it must be the case that $\beta + \varepsilon \leq 0$, which implies that $\beta < 0$. Moreover, by definition of $\tilde{K}$, $g = g_2 + \ldots + g_n$, and $g_m(x) \geq 0$ for each $x \in K_m$ and each $m = 2, \ldots, n$.

Since K_m is closed and $0 \notin K_m$, $m = 2, \ldots, n$, there exist $\alpha_m > 0$ and δ_m continuous functionals such that $\alpha_m \leq \delta_m(x) \leq \frac{-\beta}{2(n-1)}$, $x \in K_m$, $m = 2, \ldots, n$. Let $\alpha = \min\{\alpha_2, \ldots, \alpha_n\} > 0$.

Then $(g_m + \delta_m)(x) \geq \alpha > 0$ for each $m = 2, \ldots, n$ and for each $x \in K_m$. Furthermore, $\sum_{m=2}^{n}(g_m + \delta_m)(x) \leq \frac{\beta}{2}$ for all $x \in \hat{K}$.

Note that K_1 and $\hat{K}$ are isomorphic, hence $\sum_{m=2}^{n}(g_m + \delta_m)(x) \leq \frac{\beta}{2}$ for all $x \in K_1$.

Finally, for each $m = 2, \ldots, n$ let $f_m = g_m + \delta_m$. We now have all the ingredients for defining a zero-sum agreeable bet: let $f_1 = -\sum_{m=2}^{n} f_m$. Then $f_m(x) \geq \alpha > 0$ for each $m = 1, \ldots, n$ and each $x \in K_m$, and $\sum f_m = 0$.

An example I

Let $\Omega = \{\omega_1, \omega_2\}$, $N = \{1, 2\}$, $t_1 = \delta_{\{\omega_1\}}$, $t_2 = \delta_{\{\omega_2\}}$, $\mathcal{M} = \mathcal{P}(\Omega)$, and $\mathcal{M}_i = \{\emptyset, \Omega\}$ for each player $i \in N$. Then only the pair of functions

$$f_1(\omega) = \begin{cases} \alpha & \text{if } x = \omega_1, \\ -\beta & \text{if } x = \omega_2, \end{cases} \tag{5}$$

and $f_2 = -f_1$ form an agreeable bet for the type space $((\Omega, \mathcal{M}), \{\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$, for all $\alpha, \beta > 0$.
Let $\Omega_i = \Omega \times \{i\}$ for each $i \in \mathbb{R}$, yielding continuum-many copies of Ω. Let $\Omega^* = \cup_{i \in I} \Omega_i$, let $\mathcal{M}^*$ be the field generated by the sets $\{\omega\}$, for $\omega \in \Omega^*$ (i.e., the coarsest field containing the singletons of Ω^*), and let

$$t_j^*(\omega, \{x\}) = \begin{cases} t_j((\omega|_\Omega), \{x|_\Omega\}) & \text{if } \exists i \in I \text{ such that } \omega, x \in \Omega_i, \\ 0 & \text{otherwise,} \end{cases}$$

$j = 1, 2.$

An example II

Define $\mathcal{M}_j^*$ to be the field generated by Ω_i, $i \in I$, $j \in N = \{1, 2\}$. Then $T^* = ((\Omega^*, \mathcal{M}^*), \{(\Omega^*, \mathcal{M}_j^*)\}_{j\in N}, \{t_j^*\}_{j\in N})$ forms a type space.

Suppose by contradiction that there exists an agreeable bet (f_1, f_2) on T^*. Then for each $i \in I$, the functions f_1 and f_2 restricted to Ω_i must be defined on Ω_i as in Equation (5) (the specific values of α and β might depend on i).

Hence there exist $\varepsilon > 0$ such that $f_1^{-1}([\varepsilon, \infty))$ and $f_1^{-1}((-\infty, -\varepsilon))$ are both unions of infinitely many singletons, meaning that f_1 is not $\mathcal{M}^*$-integrable, hence (f_1, f_2) cannot be an agreeable bet.

As there is no agreeable bet on T^*, by our Theorem T^* admits a common prior. Indeed, it is easy to check that the probability charge on $\mathcal{M}^*$ which assigns 0 to each finite set is a common prior.

Moreover, for any $f \in B(\Omega^*, \mathcal{M}^*)$, for any finite copy of $(\Omega, \mathcal{M})$, for each $\omega' \in \Omega'$, $n \in N$

$$\left| \int f \, dt_n(\psi(\omega'), \cdot) - \int f \circ \psi \, \mathrm{d}\, t'_n(\omega') \right| = 0,$$

An example III

where $\Omega' = \cup_{i \in J} \Omega_i$, $J \subseteq I$ is an arbitrary nonempty, finite set, $\mathcal{M}' = \mathcal{M}|_{\Omega'}$ and $t'_n = t_n \circ \psi$, $i \in N$, $\psi \colon \Omega' \to \Omega$ is such that $\psi(\omega) = \omega$, $\omega \in \Omega'$.
In words, T^* has many finite, even 'perfect', approximations. Despite this, none of these finite approximations admits a common prior.

Thank you for the attention!

References I

Aumann RJ (1976) Agreeing to Disagree. The Annals of Statistics 4(6):1236–1239

Feinberg Y (2000) Characterizing Common Priors in the Form of Posteriors. Journal of Economic Theory 91:127–179

Heifetz A (2006) The positive foundation of the common prior assumption. Games and Economic Behavior 56:105–120

Lehrer E, Samet D (2014) Belief consistency and trade consistency. Games and Economic Behavior 83:165–177

Samet D (1998) Common Priors and Separation of Convex Sets. Games and Economic Behavior 24:172–174

Why Bets Must Be Bounded Away From Zero I

Our main theorem characterises the existence of common priors using agreeable bets which are not only give rise to positive expectations at each state but have expectations bounded away from zero. We show here by an example why this presumption is necessary.

There are two players, Anne and Ben. The state space and partition is the basic partition space, that is, $\Omega = \{1, 2, \ldots\}$, Anne's knowledge partition, Π^A, is given by

$$\{\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \ldots\}$$

and Ben's knowledge partition, Π^B, is given by

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots\}.$$

The epistemic events: $\mathcal{M}$ is the field generated by the singleton sets, $\mathcal{M}_i$ is the field generated by the partition Π^i, $i = A, B$. Anne's type function, t_A, is given by

Why Bets Must Be Bounded Away From Zero II

$$t_A(n, \{n\}) = \begin{cases} 1 & \text{if } n = 1 \\ \frac{1}{2} & \text{if } n \text{ is even} \\ \frac{1}{2} & \text{if } n \text{ is odd, } n > 1. \end{cases}$$

Ben's type function, t_B, is given by

$$t_B(n, \{n\}) = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd} \\ \frac{1}{2} & \text{if } n \text{ is even.} \end{cases}$$

Anne	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Ben	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Figure: The type space of the example in this section.

Why Bets Must Be Bounded Away From Zero III

Notice that the probability charge which assigns zero to each finite set is a common prior for this type space.

Suppose that f_A, with $f_B = -f_A$, is an agreeable bet, with $|f_A|$ a bounded and strictly increasing function and $f_A(n) = (-1)^{n+1}|f_A(n)|$, $n \in \Omega$. For $i = A, B$, for each $n \in \Omega$,

$$\int f_i \, \mathrm{dt_i(n, \cdot)} > 0,$$

but there does not exist $\alpha \in \mathbb{R}$ such that for each $n \in \Omega$

$$\int f_i \, \mathrm{dt_i(n, \cdot)} \geq \alpha > 0.$$

In words, this type space has an agreeable bet in sense of Lehrer and Samet (2014) but it does not have an agreeable bet in sense of our definition.

A lemma I

<u>*Lemma*</u>: Take the sets $P_1, P_2 \subseteq \ell_1 \subset \mathrm{ba}(\mathbb{N})$. Then $\overline{P_1}^{TV}$ and $\overline{P_2}^{TV}$ are strongly separable by a (norm continuous) linear functional if and only if $\overline{P_1}^*$ and $\overline{P_2}^*$ are strongly separable by a (weak* continuous) linear functional.

<u>*Proof*</u>: If: Since $\overline{P_i}^{TV} \subseteq \overline{P_1}^*$, $i = 1, 2$, if $\overline{P_1}^*$ and $\overline{P_2}^*$ are strongly separable by a linear functional, then the very same linear functional strongly separates $\overline{P_1}^{TV}$ and $\overline{P_2}^{TV}$.

Only if: Suppose that $\overline{P_1}^{TV}$ and $\overline{P_2}^{TV}$ are strongly separable by a linear function over ℓ_1, and let f be the non-trivial strongly separating linear functional, i.e, there exist $c \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$f(x) \geq c + \varepsilon \quad \text{and} \quad c - \varepsilon \geq f(y),$$

for all $x \in \overline{P_1}^{TV}$ and for all $y \in \overline{P_2}^{TV}$.

A lemma II

Since $f \in \ell_\infty$ it holds that

$$f(x) \geq c + \varepsilon \quad \text{and} \quad c - \varepsilon \geq f(y),$$

for all $x \in \overline{P_1}^*$ and for all $y \in \overline{P_2}^*$.