On Sustainable Equilibria

Hari Govindan, Rida Laraki & Lucas Pahl

Game Theory World Seminar
August, 10, 2020
The Beginning: Essays in Honor of Selten 65th Birthday

Wulf Albers · Werner Güth
Peter Hammerstein · Benny Moldovanu
Eric van Damme (Eds.)

With the help of Martin Strobel

Understanding Strategic Interaction

Essays in Honor of Reinhard Selten

Govindan, Laraki, & Pahl

Sustainable Equilibria in Culturally Familiar Games

Roger B. Myerson

J. L. Kellogg Graduate School of Management, Northwestern University, Evanston, Illinois 60208, U.S.A.

It is a pleasure to write for a volume in honor of Reinhard Selten, who has been to me both a mentor and a friend. Like many others, I am indebted to him for guidance and inspiration in many domains, from the logical study of rational behavior, to the pleasures of walking in the German countryside.

The definition of equilibrium was offered by Nash (1951) as the basic characterization of rational behavior in games, but Selten (1965, 1975) showed that the set of Nash equilibria may be too weak to characterize rational behavior in many games. That is, there may be strategic scenarios which pass the test of being Nash equilibria, but which do not fit our intuitive sense of what rational behavior should be in a game. So Selten launched the literature on refinements of Nash equilibrium, one of the great central strands of the literature of noncooperative game theory.
Desirable Criteria for Refinement

one of the great central strands of the literature of noncooperative game theory.

To evaluate a refined equilibrium concept, we may apply four criteria. First, the
definition of the refined equilibrium concept should have some intuitive appeal as a
characterization of what is rational behavior. Second, in games where we have
strong intuitions about what are the reasonable equilibria, the refined solution
concept should coincide with our intuitions, eliminating the unreasonable equilibria
but not eliminating the reasonable ones. Third, the set of refined equilibria should
be nonempty for all games to which this solution concept is supposed to be applied.
Fourth, our refined solution concept should be invariant under transformations of the
game that seem intuitively irrelevant. No solution concept in the literature has yet
fully satisfied all of these criteria (even as far as we can reach a consensus about
what is "intuitively reasonable"), and so the search for new refinements of the
equilibrium concept continues. In this essay, I want to sketch one problem area in
which more work on refinements of Nash equilibrium seems particularly needed.
Sustainable Equilibria in the Battle of the Sexes

Thus, when the Battle of the Sexes game is played as a culturally familiar game (with the roles of players 1 and 2 being distinctly identifiable across plays), we may expect that the players should have evolved a pattern of expectations which rules out the mixed equilibrium but admits either of the pure-strategy equilibria. For now, let us say informally that the two pure-strategy equilibria seem to be "sustainable" in a way the mixed equilibrium does not. This description begs the question of what kind of formal solution concept can we define to identify the "sustainable" equilibria that might be expected to persist in any strategic-form game, when it is played in a culturally familiar context. That is, how should we extend this intuitive concept of "sustainability" from the Battle of the Sexes game to a general refinement of Nash equilibrium that can be applied to any strategic-form game?

One naive approach would be to view this as a question of distinguishing
Sustainable Equilibria in the Battle of the Sexes

Myerson considers, and then dismisses existing refinements that yield the same prediction in the Battle-of-Sexes game:

(\(a_1, b_2\)) happens 50\% of the time, for example.

Thus, when the Battle of the Sexes game is played as a culturally familiar game (with the roles of players 1 and 2 being distinctly identifiable across plays), we may expect that the players should have evolved a pattern of expectations which rules out the mixed equilibrium but admits either of the pure-strategy equilibria. For now, let us say informally that the two pure-strategy equilibria seem to be "sustainable" in a way the mixed equilibrium does not. This description begs the question of what kind of formal solution concept can we define to identify the "sustainable" equilibria that might be expected to persist in any strategic-form game, when it is played in a culturally familiar context. That is, how should we extend this intuitive concept of "sustainability" from the Battle of the Sexes game to a general refinement of Nash equilibrium that can be applied to any strategic-form game?

One naive approach would be to view this as a question of distinguishing Myerson considers, and then dismisses existing refinements that yield the same prediction in the Battle-of-Sexes game:
Sustainable Equilibria in the Battle of the Sexes

Myerson considers, and then dismisses existing refinements that yield the same prediction in the Battle-of-Sexes game:

**Persistent Equilibria (Kalai-Samet): fail invariance**

Govindan, Laraki, & Pahl
Sustainable Equilibria in the Battle of the Sexes

\((u_1, u_2)\) happens 50% of the time, for example.

Thus, when the Battle of the Sexes game is played as a culturally familiar game (with the roles of players 1 and 2 being distinctly identifiable across plays), we may expect that the players should have evolved a pattern of expectations which rules out the mixed equilibrium but admits either of the pure-strategy equilibria. For now, let us say informally that the two pure-strategy equilibria seem to be "sustainable" in a way the mixed equilibrium does not. This description begs the question of what kind of formal solution concept can we define to identify the "sustainable" equilibria that might be expected to persist in any strategic-form game, when it is played in a culturally familiar context. That is, how should we extend this intuitive concept of "sustainability" from the Battle of the Sexes game to a general refinement of Nash equilibrium that can be applied to any strategic-form game?

One naive approach would be to view this as a question of distinguishing

Myerson considers, and then dismisses existing refinements that yield the same prediction in the Battle-of-Sexes game:

**Persistent Equilibria** (Kalai-Samet): fail invariance

**ESS** (Maynard-Smith): fail existence
I can think of another approach to our problem which is more purely mathematical. The generic oddness of the number of Nash equilibria (see Harsanyi, 1973) is derived from the fixed-point theorems of algebraic topology. To understand the intuition behind these theorems, it may be helpful to think about the following example. Let \( y : \mathbb{R} \to \mathbb{R} \) be any continuous function such that 
\[
\lim_{x \to +\infty} y(x) = +\infty, \quad \text{and} \quad \lim_{x \to -\infty} y(x) = -\infty.
\]
(For example, think of \( y(x) = x^3 - x \).) The graph of \( y(\bullet) \) must cross the x-axis (where \( y = 0 \)) an odd number of times because, as we move from negative \( x \) to positive \( x \), there must be one more crossing up from below than crossings down from above. That is, each crossing of the x-axis can be assigned an index value, +1 for each crossing up from below (where \( y(x) \) is locally increasing in \( x \)) and -1 for each crossing down from above (where \( y(x) \) is locally decreasing), and then the sum of the indexes of all crossings must be +1. (Tangencies where the graph does not cross
Illustration: odd number of crosses, sum of indices = +1
Link with Fixed Point and Equilibria
This mathematical heuristic suggests that we might look for some way of assigning index numbers to Nash equilibria such that, for all generic games, each equilibrium can be assigned the index +1 or -1, and the sum of the indexes of all equilibria is +1. If any such theory were applied to the Battle of Sexes game, the symmetry of the two pure strategy equilibria should guarantee that they must have the same index number, and so the pure-strategy equilibria of this game would have index +1 and the mixed equilibrium must have index -1. Thus, generalizing this heuristic, we may wonder whether the mathematics of differential topology might be able to give us some sophisticated index function such that the sustainable equilibria have index +1 and the unsustainable equilibria have index -1.
Myerson’s Conjecture

This mathematical heuristic suggests that we might look for some way of assigning index numbers to Nash equilibria such that, for all generic games, each equilibrium can be assigned the index +1 or -1, and the sum of the indexes of all equilibria is +1. If any such theory were applied to the Battle of Sexes game, the symmetry of the two pure strategy equilibria should guarantee that they must have the same index number, and so the pure-strategy equilibria of this game would have index +1 and the mixed equilibrium must have index -1. Thus, generalizing this heuristic, we may wonder whether the mathematics of differential topology might be able to give us some sophisticated index function such that the sustainable equilibria have index +1 and the unsustainable equilibria have index -1.

Myerson was seemingly unaware of the existence at that moment of an index theory for Nash equilibria:

Index of equilibria for economists: THE BOOK

Andrew McLennan

Advanced Fixed Point Theory for Economics

Govindan, Laraki, & Pahl
Myerson’s Conclusion

In this paper, I have discussed the importance of trying to define some formal concept of "sustainability" of equilibria, to distinguish equilibria that are likely to be played in culturally familiar settings. I have not offered any precise formal definition of "sustainability," however. Instead, I have tried to describe an open problem, which I have pondered for long time. I have occasionally found promising new approaches to this problem, but so far none have led to the kind of simple and appealing formulation that I feel intuitively must be out there somewhere. Still I have hope that tomorrow someone may have the conceptual breakthrough that can help to fill this important gap in the game-theory literature.
Myerson’s Conclusion

In this paper, I have discussed the importance of trying to define some formal concept of "sustainability" of equilibria, to distinguish equilibria that are likely to be played in culturally familiar settings. I have not offered any precise formal definition of "sustainability," however. Instead, I have tried to describe an open problem, which I have pondered for long time. I have occasionally found promising new approaches to this problem, but so far none have led to the kind of simple and appealing formulation that I feel intuitively must be out there somewhere. Still I have hope that tomorrow someone may have the conceptual breakthrough that can help to fill this important gap in the game-theory literature.

Hofbauer (2000) formulated a precise definition of sustainability for generic games and some conjectures.
Hofbauer Formulation of Myerson’s Criteria

- (A1) **Strict** Nash equilibria are sustainable.
Hofbauer Formulation of Myerson’s Criteria

- (A1) **Strict** Nash equilibria are sustainable.
- (A2) Battle of sexes: **only strict equilibria are sustainable.**
Hofbauer Formulation of Myerson’s Criteria

- (A1) **Strict** Nash equilibria are sustainable.
- (A2) Battle of sexes: only **strict** equilibria are sustainable.
- (A3) If a game has a **unique** equilibrium, it is sustainable.
Hofbauer Formulation of Myerson’s Criteria

- (A1) **Strict** Nash equilibria are sustainable.
- (A2) Battle of sexes: **only strict equilibria are sustainable**.
- (A3) If a game has a **unique equilibrium**, it is sustainable.
- (A4) Every **generic game** has a sustainable equilibrium.
Hofbauer Formulation of Myerson’s Criteria

- (A1) **Strict** Nash equilibria are sustainable.
- (A2) Battle of sexes: only strict equilibria are sustainable.
- (A3) If a game has a **unique equilibrium**, it is sustainable.
- (A4) Every **generic game** has a sustainable equilibrium.
- (A5) Sustainable equilibria are **invariant** under addition or removal of inferior replies.
Hofbauer Formulation of Myerson’s Criteria

- (A1) **Strict** Nash equilibria are sustainable.
- (A2) Battle of sexes: only strict equilibria are sustainable.
- (A3) If a game has a unique equilibrium, it is sustainable.
- (A4) Every **generic game** has a sustainable equilibrium.
- (A5) Sustainable equilibria are **invariant** under addition or removal of inferior replies.

**A5 is a form of independence of irrelevant alternative.**
Hofbauer Formulation of Myerson’s Criteria

- (A1) **Strict** Nash equilibria are sustainable.
- (A2) Battle of sexes: only strict equilibria are sustainable.
- (A3) If a game has a unique equilibrium, it is sustainable.
- (A4) Every *generic game* has a sustainable equilibrium.
- (A5) Sustainable equilibria are invariant under addition or removal of inferior replies.

**A5 is a form of independence of irrelevant alternative.**

It is a very strong axiom as, combined with A3, they imply A1.
Hofbauer Definition of Sustainability

Hofbauer defines an equivalence relation among pairs \((G, \sigma)\) where \(G\) is a game and \(\sigma\) is an equilibrium of \(G\).

\[(G, \sigma) \sim (\hat{G}, \hat{\sigma})\] if \(\sigma = \hat{\sigma}\) (up to a relabelling) and the restriction of \(G\) and \(\hat{G}\) to the best replies to \(\sigma\) and \(\hat{\sigma}\), resp., are the same game (up to a relabelling).

By A5 (IIA) and A3 (uniqueness = sustainability):

If an equilibrium \(\sigma\) of a game \(G\) is unique in an equivalent pair \((\hat{G}, \hat{\sigma})\), it must be sustainable.

Hofbauer cleverly combined A5 & A3 with minimality:

\[\text{An equilibrium of a game } G \text{ is sustainable iff it is the unique equilibrium in an equivalent pair.}\]
Hofbauer Definition of Sustainability

- Hofbauer defines an equivalence relation among pairs \((G, \sigma)\) where \(G\) is a game and \(\sigma\) is an equilibrium of \(G\).
Hofbauer Definition of Sustainability

- Hofbauer defines an equivalence relation among pairs \((G, \sigma)\) where \(G\) is a game and \(\sigma\) is an equilibrium of \(G\).

- \((G, \sigma) \sim (\hat{G}, \hat{\sigma})\) if \(\sigma = \hat{\sigma}\) (up to a relabelling) and the restriction of \(G\) and \(\hat{G}\) to the best replies to \(\sigma\) and \(\hat{\sigma}\), resp., are the same game (up to a relabelling).
Hofbauer Definition of Sustainability

Hofbauer defines an equivalence relation among pairs \((G, \sigma)\) where \(G\) is a game and \(\sigma\) is an equilibrium of \(G\).

\((G, \sigma) \sim (\hat{G}, \hat{\sigma})\) if \(\sigma = \hat{\sigma}\) (up to a relabelling) and the restriction of \(G\) and \(\hat{G}\) to the best replies to \(\sigma\) and \(\hat{\sigma}\), resp., are the same game (up to a relabelling).

By A5 (IIA) and A3 (uniqueness \(\Rightarrow\) sustainability): If an equilibrium \(\sigma\) of a game \(G\) is unique in an equivalent pair \((\hat{G}, \hat{\sigma})\), it must be sustainable.
Hofbauer Definition of Sustainability

- Hofbauer defines **an equivalence relation** among pairs \((G, \sigma)\) where \(G\) is a game and \(\sigma\) is an equilibrium of \(G\).

- \((G, \sigma) \sim (\hat{G}, \hat{\sigma})\) if \(\sigma = \hat{\sigma}\) (up to a relabelling) and the **restriction of \(G\) and \(\hat{G}\)** to the best replies to \(\sigma\) and \(\hat{\sigma}\), resp., are the same game (up to a relabelling).

- By A5 (IIA) and A3 (uniqueness \(\Rightarrow\) sustainability): If an equilibrium \(\sigma\) of a game \(G\) is unique in an equivalent pair \((\hat{G}, \hat{\sigma})\), it must be sustainable.

Hofbauer cleverly combined A5 & A3 with minimality:
Hofbauer Definition of Sustainability

- Hofbauer defines an equivalence relation among pairs \((G, \sigma)\) where \(G\) is a game and \(\sigma\) is an equilibrium of \(G\).

- \((G, \sigma) \sim (\hat{G}, \hat{\sigma})\) if \(\sigma = \hat{\sigma}\) (up to a relabelling) and the restriction of \(G\) and \(\hat{G}\) to the best replies to \(\sigma\) and \(\hat{\sigma}\), resp., are the same game (up to a relabelling).

- By A5 (IIA) and A3 (uniqueness \(\Rightarrow\) sustainability): If an equilibrium \(\sigma\) of a game \(G\) is unique in an equivalent pair \((\hat{G}, \hat{\sigma})\), it must be sustainable.

Hofbauer cleverly combined A5 & A3 with minimality:

- An equilibrium of a game \(G\) is sustainable iff it is the unique equilibrium in an equivalent pair.
Battle of the Sexes

3 Nash equilibria: 2 strict $\sigma = (t, l)$ and $\theta = (b, r)$, and 1 mixed.

\[ G = \begin{array}{c|cc}
 & l & r \\
\hline
t & (3, 2) & (0, 0) \\
b & (0, 0) & (2, 3) \\
\end{array} \]

Hence, the strict equilibrium $\sigma$ is sustainable in $G$.

The mixed equilibrium is not sustainable (prove it?).

This is in line with Myerson requirements A2.
Battle of the Sexes

3 Nash equilibria: 2 strict $\sigma = (t, l)$ and $\theta = (b, r)$, and 1 mixed.

$$G = \begin{array}{cc}
  l & r \\
  t & (3,2) & (0,0) \\
  b & (0,0) & (2,3)
\end{array}$$

By adding two strategies, $\sigma$ is the unique equilibrium of $\hat{G}$:

$$\hat{G} = \begin{array}{ccc}
  l & r & y \\
  t & (3,2) & (0,0) & (0,1) \\
  b & (0,0) & (2,3) & (−2,4) \\
  x & (1,0) & (4,−2) & (−1,−1)
\end{array}$$

Hence, the strict equilibrium $\sigma$ is sustainable in $G$.

The mixed equilibrium is not sustainable (prove it?).

This is in line with Myerson requirements A2.
Battle of the Sexes

3 Nash equilibria: 2 strict $\sigma = (t, l)$ and $\theta = (b, r)$, and 1 mixed.

\[
G = \begin{array}{ccc}
& l & r \\
t & (3, 2) & (0, 0) \\
& (0, 0) & (2, 3) \\
\end{array}
\]

By adding two strategies, $\sigma$ is the unique equilibrium of $\hat{G}$:

\[
\hat{G} = \begin{array}{ccc}
& l & r & y \\
t & (3, 2) & (0, 0) & (0, 1) \\
& (0, 0) & (2, 3) & (\text{?}, \text{?}) \\
x & (1, 0) & (4, -2) & (\text{?}, \text{?}) \\
\end{array}
\]

▶ Hence, the strict equilibrium $\sigma$ is sustainable in $G$. 
**Battle of the Sexes**

3 Nash equilibria: 2 strict $\sigma = (t, l)$ and $\theta = (b, r)$, and 1 mixed.

$$G = \begin{array}{c|cc}
  & l & r \\
\hline
 t & (3, 2) & (0, 0) \\
 b & (0, 0) & (2, 3) \\
\end{array}$$

By adding two strategies, $\sigma$ is the unique equilibrium of $\hat{G}$:

$$\hat{G} = \begin{array}{c|ccc}
  & l & r & y \\
\hline
 t & (3, 2) & (0, 0) & (0, 1) \\
 b & (0, 0) & (2, 3) & (\text{---} -2, 4) \\
 x & (1, 0) & (4, -2) & (\text{---} -1, -1) \\
\end{array}$$

- Hence, the strict equilibrium $\sigma$ is sustainable in $G$.
- The mixed equilibrium is not sustainable (prove it?).
Battle of the Sexes

3 Nash equilibria: 2 strict $\sigma = (t, l)$ and $\theta = (b, r)$, and 1 mixed.

$$\begin{array}{c|cc}
 & l & r \\
\hline
 t & (3, 2) & (0, 0) \\
b & (0, 0) & (2, 3) \\
\end{array}$$

By adding two strategies, $\sigma$ is the unique equilibrium of $\hat{G}$:

$$\begin{array}{c|ccc}
 & l & r & y \\
\hline
 t & (3, 2) & (0, 0) & (0, 1) \\
b & (0, 0) & (2, 3) & (-2, 4) \\
x & (1, 0) & (4, -2) & (-1, -1) \\
\end{array}$$

▶ Hence, the strict equilibrium $\sigma$ is sustainable in $G$.
▶ The mixed equilibrium is not sustainable (prove it?).
▶ This is in line with Myerson requirements A2.
**A Sustainable Mixed Equilibrium**

$G$ has 7 equilibria, all symmetric: 3 strict, 3 mixed (players randomise between 2 strategies), and 1 completely mixed.

$G_1 = \begin{array}{ccc}
  & l & m & r \\
  t & (10, 10) & (0, 0) & (0, 0) \\
  m & (0, 0) & (10, 10) & (0, 0) \\
  b & (0, 0) & (0, 0) & (10, 10) \\
\end{array}$
A Sustainable Mixed Equilibrium

$G$ has 7 equilibria, all symmetric: 3 strict, 3 mixed (players randomise between 2 strategies), and 1 completely mixed.

\[
G_1 = \begin{array}{ccc}
  t & l & m & r \\
  & (10, 10) & (0, 0) & (0, 0) \\
  m & (0, 0) & (10, 10) & (0, 0) \\
  b & (0, 0) & (0, 0) & (10, 10) \\
\end{array}
\]

The 3 strict equilibria and the completely mixed are sustanables, as $\hat{G}$ shows (von Schemde & von Stengel).
A Sustainable Mixed Equilibrium

\( G \) has 7 equilibria, all symmetric: 3 strict, 3 mixed (players randomise between 2 strategies), and 1 completely mixed.

\[
G_1 = \begin{pmatrix}
    t & l \quad (10, 10) & m \quad (0, 0) & r \quad (0, 0) \\
    m & (0, 0) \quad (10, 10) & (0, 0) \\
    b & (0, 0) \quad (0, 0) \quad (10, 10)
\end{pmatrix}
\]

The 3 strict equilibria and the completely mixed are sustainables, as \( \hat{G} \) shows (von Schemde & von Stengel).

\[
\hat{G}_1 = \begin{pmatrix}
    t & l \quad (10, 10) & m \quad (0, 0) & r \quad (0, 0) & x \quad (0, 11) & y \quad (10, 5) & z \quad (0, -10) \\
    m & (0, 0) \quad (10, 10) & (0, 0) & (0, -10) & (0, 11) & (10, 5) \\
    b & (0, 0) \quad (0, 0) \quad (10, 10) \quad (10, 5) & (0, -10) & (0, 11)
\end{pmatrix}
\]
A Sustainable Mixed Equilibrium

$G$ has 7 equilibria, all symmetric: 3 strict, 3 mixed (players randomise between 2 strategies), and 1 completely mixed.

\[
G_1 = \begin{array}{ccc}
t & l & m & r \\
(10, 10) & (0, 0) & (0, 0) \\
(0, 0) & (10, 10) & (0, 0) \\
(0, 0) & (0, 0) & (10, 10) \\
\end{array}
\]

The 3 strict equilibria and the completely mixed are sustainables, as $\hat{G}$ shows (von Schemde & von Stengel).

\[
\hat{G}_1 = \begin{array}{ccccccc}
t & l & m & r & x & y & z \\
(10, 10) & (0, 0) & (0, 0) & (0, 11) & (10, 5) & (0, -10) \\
(0, 0) & (10, 10) & (0, 0) & (0, -10) & (0, 11) & (10, 5) \\
(0, 0) & (0, 0) & (10, 10) & (10, 5) & (0, -10) & (0, 11) \\
\end{array}
\]

3 remaining mixed equilibria are not sustainable (prove it?)

Govindan, Laraki, & Pahl
Notations

- Set of players is $\mathcal{N} = \{1, \ldots, N\}$. 
Notations

- Set of players is \( \mathcal{N} = \{1, \ldots, N\} \).
- \( \forall n \), a finite set \( S_n \) of pure strategies.
Notations

- Set of players is $\mathcal{N} = \{1, \ldots, N\}$.
- $\forall n$, a finite set $S_n$ of pure strategies.
- The set of pure strategy profiles $S \equiv \prod_{n \in \mathcal{N}} S_n$. 
Notations

- **Set of players** is $\mathcal{N} = \{1, \ldots, N\}$.
- **∀n**: a finite set $S_n$ of pure strategies.
- **The set of pure strategy profiles** $S \equiv \prod_{n \in \mathcal{N}} S_n$.
- $\Sigma_n$ is player $n$’s set of mixed strategies.
Notations

- Set of players is $\mathcal{N} = \{1, \ldots, N\}$.
- $\forall n$, a finite set $S_n$ of pure strategies.
- The set of pure strategy profiles $S \equiv \prod_{n \in \mathcal{N}} S_n$.
- $\Sigma_n$ is player $n$’s set of mixed strategies.
- The set of mixed strategy profiles $\Sigma \equiv \prod_{n \in \mathcal{N}} \Sigma_n$.
Notations

- Set of players is $\mathcal{N} = \{1, \ldots, N\}$.
- $\forall n$, a finite set $S_n$ of pure strategies.
- The set of pure strategy profiles $S \equiv \prod_{n \in \mathcal{N}} S_n$.
- $\Sigma_n$ is player $n$’s set of mixed strategies.
- The set of mixed strategy profiles $\Sigma \equiv \prod_{n \in \mathcal{N}} \Sigma_n$.
- For each $n$, let $S_{-n} = \prod_{m \neq n} S_m$ and $\Sigma_{-n} = \prod_{m \neq n} \Sigma_m$. 
Notations

- Set of players is $\mathcal{N} = \{1, \ldots, N\}$.
- $\forall n$, a finite set $S_n$ of pure strategies.
- The set of pure strategy profiles $S \equiv \prod_{n \in \mathcal{N}} S_n$.
- $\Sigma_n$ is player $n$’s set of mixed strategies.
- The set of mixed strategy profiles $\Sigma \equiv \prod_{n \in \mathcal{N}} \Sigma_n$.
- For each $n$, let $S_{-n} = \prod_{m \neq n} S_m$ and $\Sigma_{-n} = \prod_{m \neq n} \Sigma_m$.
- $G : S \rightarrow \mathbb{R}^\mathcal{N}$ is the payoff vector function.
Notations

- Set of players is $\mathcal{N} = \{1, \ldots, N\}$.
- $\forall n$, a finite set $S_n$ of pure strategies.
- The set of pure strategy profiles $S \equiv \prod_{n \in \mathcal{N}} S_n$.
- $\Sigma_n$ is player $n$’s set of mixed strategies.
- The set of mixed strategy profiles $\Sigma \equiv \prod_{n \in \mathcal{N}} \Sigma_n$.
- For each $n$, let $S_{-n} = \prod_{m \neq n} S_m$ and $\Sigma_{-n} = \prod_{m \neq n} \Sigma_m$.
- $G : S \to \mathbb{R}^\mathcal{N}$ is the payoff vector function.
- $\Gamma \equiv \mathbb{R}^{\mathcal{N} \times S}$: is the space of games as payoffs vary.
Notations

- Set of players is $\mathcal{N} = \{1, \ldots, N\}$.
- $\forall n$, a finite set $S_n$ of pure strategies.
- The set of pure strategy profiles $S \equiv \prod_{n \in \mathcal{N}} S_n$.
- $\Sigma_n$ is player $n$’s set of mixed strategies.
- The set of mixed strategy profiles $\Sigma \equiv \prod_{n \in \mathcal{N}} \Sigma_n$.
- For each $n$, let $S_{-n} = \prod_{m \neq n} S_m$ and $\Sigma_{-n} = \prod_{m \neq n} \Sigma_m$.
- $G : S \to \mathbb{R}^\mathcal{N}$ is the payoff vector function.
- $\Gamma \equiv \mathbb{R}^{\mathcal{N} \times S}$: is the space of games as payoffs vary.
- $\mathcal{E} = \{(G, \sigma) \in \Gamma \times \Sigma \mid \sigma$ is a Nash equilibrium of $G \}$ is the Nash equilibrium correspondence.
Index of Equilibria

- Let $G$ be a game and $U$ be a neighbourhood of $\Sigma$ in $\mathbb{R}^{N|S|}$. 

 Govindan, Laraki, & Pahl
Index of Equilibria

- Let $G$ be a game and $U$ be a neighbourhood of $\Sigma$ in $\mathbb{R}^{N|S|}$.

- Let $f = f_G : U \rightarrow \Sigma$ be a differentiable map (continuously dependent on $G$) whose fixed points are the equilibria of $G$. 

Govindan, Laraki, & Pahl
Index of Equilibria

- Let $G$ be a game and $U$ be a neighbourhood of $\Sigma$ in $\mathbb{R}^{N|S|}$.

- Let $f = f_G : U \to \Sigma$ be a differentiable map (continuously dependent on $G$) whose fixed points are the equilibria of $G$.

- Nash equilibria of $G$ are zeros of $\sigma \to d(\sigma) := \sigma - f(\sigma)$.

Let $\sigma$ is regular if the Jacobian of $d$ at $\sigma$ is nonsingular.

A game is regular if all its Nash equilibria are regular.

$\text{Index}(\sigma) = \pm 1 = \text{sign of the determinant of the Jacobian}$.

$\text{Index}$ is independent on which map $f_G$ is used.

Govindan, Laraki, & Pahl
Index of Equilibria

- Let $G$ be a game and $U$ be a neighbourhood of $\Sigma$ in $\mathbb{R}^{N|S|}$.
- Let $f = f_G : U \to \Sigma$ be a differentiable map (continuously dependent on $G$) whose fixed points are the equilibria of $G$.
- Nash equilibria of $G$ are zeros of $\sigma \to d(\sigma) := \sigma - f(\sigma)$.
- $\sigma$ is regular if the Jacobian of $d$ at $\sigma$ is nonsingular.
Index of Equilibria

- Let $G$ be a game and $U$ be a neighbourhood of $\Sigma$ in $\mathbb{R}^{N|S|}$.
- Let $f = f_G : U \rightarrow \Sigma$ be a differentiable map (continuously dependent on $G$) whose fixed points are the equilibria of $G$.
- Nash equilibria of $G$ are zeros of $\sigma \rightarrow d(\sigma) := \sigma - f(\sigma)$.
- $\sigma$ is regular if the Jacobian of $d$ at $\sigma$ is nonsingular.
- A game is regular if all its Nash equilibria are regular.
Index of Equilibria

Let $G$ be a game and $U$ be a neighbourhood of $\Sigma$ in $\mathbb{R}^{N|S|}$.

Let $f = f_G : U \to \Sigma$ be a differentiable map (continuously dependent on $G$) whose fixed points are the equilibria of $G$.

Nash equilibria of $G$ are zeros of $\sigma \to d(\sigma) := \sigma - f(\sigma)$.

$\sigma$ is regular if the Jacobian of $d$ at $\sigma$ is nonsingular.

A game is regular if all its Nash equilibria are regular.

$\text{Index}(\sigma) = \pm 1 = \text{sign of the determinant of the Jacobian}$. 
Index of Equilibria

- Let $G$ be a game and $U$ be a neighbourhood of $\Sigma$ in $\mathbb{R}^{N|S|}$.

- Let $f = f_G : U \to \Sigma$ be a differentiable map (continuously dependent on $G$) whose fixed points are the equilibria of $G$.

- Nash equilibria of $G$ are zeros of $\sigma \to d(\sigma) := \sigma - f(\sigma)$.

- $\sigma$ is **regular** if the Jacobian of $d$ at $\sigma$ is nonsingular.

- A game is regular if all its Nash equilibria are regular.

- $\text{Index}(\sigma) = \pm 1 = \text{sign of the determinant of the Jacobian}$.

- **Index is independent on which map** $f_G$ **is used.**
  (Demichelis & Germano)
Hofbauer-Myerson conjecture

Hofbauer-Myerson conjecture: A regular equilibrium is sustainable if and only if it has index +1.
Hofbauer-Myerson conjecture

Hofbauer-Myerson conjecture: A regular equilibrium is sustainable if and only if it has index +1.

Hofbauer-Myerson conjecture

Hofbauer-Myerson conjecture: A regular equilibrium is sustainable if and only if it has index $+1$.


- We prove the Hofbauer-Myerson conjecture for all $N$-player games using algebraic topology.
Hofbauer-Myerson conjecture

Hofbauer-Myerson conjecture: A regular equilibrium is sustainable if and only if it has index +1.

- We prove the Hofbauer-Myerson conjecture for all $N$-player games using algebraic topology.
- **Corollary 1**: since the sum of the indices of equilibria is +1, any regular game has a sustainable equilibrium.
Hofbauer-Myerson conjecture

**Hofbauer-Myerson conjecture**: A regular equilibrium is sustainable if and only if it has index $+1$.


- We prove the Hofbauer-Myerson conjecture for all $N$-player games using algebraic topology.

- **Corollary 1**: since the sum of the indices of equilibria is $+1$, any regular game has a sustainable equilibrium.

- **Corollary 2**: Since the set of regular games is open and dense, almost every game has a sustainable equilibrium.
Hofbauer-Myerson conjecture

Hofbauer-Myerson conjecture: A regular equilibrium is sustainable if and only if it has index +1.


▶ We prove the Hofbauer-Myerson conjecture for all N-player games using algebraic topology.

▶ Corollary 1: since the sum of the indices of equilibria is +1, any regular game has a sustainable equilibrium.

▶ Corollary 2: Since the set of regular games is open and dense, almost every game has a sustainable equilibrium.

This implies Myerson requirements A1, A2, A4 and A5.
Hofbauer-Myerson conjecture

Hofbauer-Myerson conjecture: A regular equilibrium is sustainable if and only if it has index +1.


- We prove the Hofbauer-Myerson conjecture for all $N$-player games using algebraic topology.

- Corollary 1: since the sum of the indices of equilibria is +1, any regular game has a sustainable equilibrium.

- Corollary 2: Since the set of regular games is open and dense, almost every game has a sustainable equilibrium.

This implies Myerson requirements A1, A2, A4 and A5.

As our proof extends to isolated equilibria, we obtain A3 (because a unique equilibrium is isolated and has index +1).
Proof: the Easy Direction

Let $\sigma$ be a regular equilibrium of game $G$. 
Proof: the Easy Direction

Let $\sigma$ be a regular equilibrium of game $G$.

- Let $(G, \sigma) \sim (\hat{G}, \sigma)$ and $\sigma =$ unique equilibrium of $\hat{G}$. 

Govindan, Laraki, & Pahl
Proof: the Easy Direction

Let \( \sigma \) be a regular equilibrium of game \( G \).

- Let \( (G, \sigma) \sim (\hat{G}, \sigma) \) and \( \sigma \) = unique equilibrium of \( \hat{G} \).
- Let \( G^* \) be obtained from \( G \) by deleting inferior replies to \( \sigma \).
Proof: the Easy Direction

Let $\sigma$ be a regular equilibrium of game $G$.

- Let $(G, \sigma) \sim (\hat{G}, \sigma)$ and $\sigma =$ unique equilibrium of $\hat{G}$.
- Let $G^*$ be obtained from $G$ by deleting inferior replies to $\sigma$.
- It follows from a property of the index that:
  - the index of $\sigma$ in $G = \text{the index of } \sigma \text{ in } G^*$.
Proof: the Easy Direction

Let \( \sigma \) be a regular equilibrium of game \( G \).

- Let \((G, \sigma) \sim (\hat{G}, \sigma)\) and \(\sigma = \text{unique equilibrium of } \hat{G}\).
- Let \(G^*\) be obtained from \(G\) by deleting inferior replies to \(\sigma\).
- It follows from a property of the index that:
  the index of \(\sigma\) in \(G\) = the index of \(\sigma\) in \(G^*\).
- \(G^*\) is also obtained from \(\hat{G}\) by deleting inferior replies.
Proof: the Easy Direction

Let $\sigma$ be a regular equilibrium of game $G$.

- Let $(G, \sigma) \sim (\hat{G}, \sigma)$ and $\sigma = \text{unique equilibrium of } \hat{G}$.
- Let $G^*$ be obtained from $G$ by deleting inferior replies to $\sigma$.
- It follows from a property of the index that: the index of $\sigma$ in $G = \text{the index of } \sigma \text{ in } G^*$.
- $G^*$ is also obtained from $\hat{G}$ by deleting inferior replies.
- Therefore, the index of $\sigma$ in $G^* = \text{index of } \sigma \text{ in } \hat{G}$. 

Govindan, Laraki, & Pahl
Proof: the Easy Direction

Let $\sigma$ be a regular equilibrium of game $G$.

- Let $(G, \sigma) \sim (\hat{G}, \sigma)$ and $\sigma = \text{unique equilibrium of } \hat{G}$.
- Let $G^*$ be obtained from $G$ by deleting inferior replies to $\sigma$.
- It follows from a property of the index that: the index of $\sigma$ in $G = \text{the index of } \sigma \text{ in } G^*$.
- $G^*$ is also obtained from $\hat{G}$ by deleting inferior replies.
- Therefore, the index of $\sigma$ in $G^* = \text{index of } \sigma \text{ in } \hat{G}$.
- As $\sigma$ is the unique equilibrium of $\hat{G}$: the index of $\sigma$ in $\hat{G} = +1$.  

Govindan, Laraki, & Pahl
Difficult Direction 1: Index = Degree

The projection map $\Pi$ projects a pair $(G, \sigma)$ in the equilibrium correspondence $\mathcal{E}$ to the game $G \in \mathcal{G}$. 

Govindan, Laraki, & Pahl
Difficult Direction 1: Index=Degree

The projection map $\Pi$ projects a pair $(G, \sigma)$ in the equilibrium correspondence $\mathcal{E}$ to the game $G \in \mathcal{G}$. 

Diagram:

- $\mathcal{E}$ is shown as a curve.
- The projection map $\Pi$ is indicated by an arrow pointing from $\mathcal{E}$ to the game $G$.
- The degree of an equilibrium $\sigma$ of a game $G$ is defined as the local orientation of the projection map $\Pi$ at $(G, \sigma)$.

Govindan, Laraki, & Pahl
Difficult Direction 1: Index=Degree

The projection map $\Pi$ projects a pair $(G, \sigma)$ in the equilibrium correspondence $\mathcal{E}$ to the game $G \in \mathcal{G}$.

**Definition:** The degree of an equilibrium $\sigma$ of a game $G$ = the local orientation of projection map $\Pi$ at $(G, \sigma)$.
Difficult Direction 1: Index = Degree

The projection map $\Pi$ projects a pair $(G, \sigma)$ in the equilibrium correspondence $\mathcal{E}$ to the game $G \in \mathcal{G}$.

**Definition:** The degree of an equilibrium $\sigma$ of a game $G$ = the local orientation of projection map $\Pi$ at $(G, \sigma)$.

**Theorem Govindan & Wilson (2005):** Degree = Index

Govindan, Laraki, & Pahl
Difficult Direction 2: Use Hopf Extension Theorem

Let $G$ be a regular game and $\sigma$ a +1 equilibrium.
Difficult Direction 2: Use Hopf Extension Theorem

Let $G$ be a regular game and $\sigma$ a $+1$ equilibrium.

- Let $f$ be the better reply map $f$ defined in Nash’s PhD whose fixed points are the equilibria of $G$. 

Hopf Theorem

Let $W$ be a compact, connected, oriented $k+1$ dimensional manifold with boundary, and let $f: \partial W \rightarrow S^k$ be a smooth map. Then $f$ extends to a globally defined map $F: W \rightarrow S^k$, with $F = f$, if and only if the degree of $f$ is zero.
Difficult Direction 2: Use Hopf Extension Theorem

Let $G$ be a regular game and $\sigma$ a $+1$ equilibrium.

- Let $f$ be the better reply map $f$ defined in Nash’s PhD whose fixed points are the equilibria of $G$.

- Since the sum of degrees over all equilibria is $+1$, the sum of degrees over all equilibria other than $\sigma$ is zero.
Difficult Direction 2: Use Hopf Extension Theorem

Let $G$ be a regular game and $\sigma$ a $+1$ equilibrium.

- Let $f$ be the better reply map $f$ defined in Nash’s PhD whose fixed points are the equilibria of $G$.
- Since the sum of degrees over all equilibria is $+1$, the sum of degrees over all equilibria other than $\sigma$ is zero.
- This implies that we can alter $f$ outside a neighbourhood $U$ of $\sigma$ so that the new map $f^0$ has only one fixed point: $\sigma$. 

Govindan, Laraki, & Pahl
Difficult Direction 2: Use Hopf Extension Theorem

Let $G$ be a regular game and $\sigma$ a $+1$ equilibrium.

- Let $f$ be the better reply map $f$ defined in Nash’s PhD whose fixed points are the equilibria of $G$.

- Since the sum of degrees over all equilibria is $+1$, the sum of degrees over all equilibria other than $\sigma$ is zero.

- This implies that we can alter $f$ outside a neighbourhood $U$ of $\sigma$ so that the new map $f^0$ has only one fixed point: $\sigma$.

- The possibility of such a construction follows from a deep result in differential topology: Hopf Extension Theorem.
Difficult Direction 2: Use Hopf Extension Theorem
Let $G$ be a regular game and $\sigma$ a $+1$ equilibrium.

- Let $f$ be the better reply map $f$ defined in Nash’s PhD whose fixed points are the equilibria of $G$.
- Since the sum of degrees over all equilibria is $+1$, the sum of degrees over all equilibria other than $\sigma$ is zero.
- This implies that we can alter $f$ outside a neighbourhood $U$ of $\sigma$ so that the new map $f^0$ has only one fixed point: $\sigma$.
- The possibility of such a construction follows from a deep result in differential topology: Hopf Extension Theorem.

**Hopf Theorem** Let $W$ be a compact, connected, oriented $k+1$ dimensional manifold with boundary, and let $f : \partial W \to S^k$ be a smooth map. Then $f$ extends to a globally defined map $F : W \to S^k$, with $F = f$, if and only if the degree of $f$ is zero.
Illustrative Example

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>(1, 1)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>$R$</td>
<td>(0, 0)</td>
<td>(1, 1)</td>
</tr>
</tbody>
</table>

For simplicity, focus on symmetric strategies. A symmetric profile is represented by a number $x \in [0, 1]$. Two strict equilibria $x = 1$ and $x = 0$ (index=degree= +1); one mixed $x = 1/2$ (index=degree − 1). The Nash map $f$ of this game is:

\[
f(x) = \begin{cases} 
    x & \text{if } x \in [0, 1/2] \\
    1 - 2x^2 + x & \text{if } x \in (1/2, 1]
\end{cases}
\]
Illustrative Example

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>(1,1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>R</td>
<td>(0,0)</td>
<td>(1,1)</td>
</tr>
</tbody>
</table>

- For simplicity, focus on symmetric strategies.
Illustrative Example

\[
\begin{array}{cc}
L & R \\
L & (1,1) & (0,0) \\
R & (0,0) & (1,1) \\
\end{array}
\]

- For simplicity, focus on symmetric strategies.
- A symmetric profile is represented by a number \( x \in [0,1] \).
Illustrative Example

\[
\begin{array}{c|cc}
 & L & R \\
\hline
L & (1, 1) & (0, 0) \\
R & (0, 0) & (1, 1) \\
\end{array}
\]

- For simplicity, focus on symmetric strategies.
- A **symmetric** profile is represented by a number \( x \in [0, 1] \).
- Two strict equilibria \( x = 1 \) and \( x = 0 \) (index=degree= +1); one mixed \( x = 1/2 \) (index=degree = -1).
Illustrative Example

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>(1, 1)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>R</td>
<td>(0, 0)</td>
<td>(1, 1)</td>
</tr>
</tbody>
</table>

- For simplicity, focus on symmetric strategies.
- A **symmetric** profile is represented by a number $x \in [0, 1]$.
- Two strict equilibria $x = 1$ and $x = 0$ (index=degree= +1); one mixed $x = 1/2$ (index=degree −1).
- The Nash map $f$ of this game is:
Illustrative Example

\[\begin{array}{cc}
L & R \\
L & (1, 1) & (0, 0) \\
R & (0, 0) & (1, 1)
\end{array}\]

- For simplicity, focus on symmetric strategies.
- A **symmetric** profile is represented by a number \(x \in [0, 1]\).
- Two strict equilibria \(x = 1\) and \(x = 0\) (index=degree= +1); one mixed \(x = 1/2\) (index=degree = -1).
- The Nash map \(f\) of this game is:

\[
f(x) \equiv \begin{cases} 
\frac{x}{1-2x^2+x} & \text{if } x \in [0, 1/2] \\
\frac{x-2x^2+3x-1}{1-2x^2+3x-1} & \text{if } x \in (1/2, 1] 
\end{cases}
\]
Illustration of Hopf Extension Theorem

Figure: Graphs of $f$ (black) and $f^0$ (green)
Illustration of Hopf Extension Theorem

Figure: Graphs of $f$ (black) and $f^0$ (green)
Difficult Direction 3: add payoff-irrelevant strategies

We have a map $f^0$ with no fixed points other than $\sigma$. 

Govindan, Laraki, & Pahl
Difficult Direction 3: add payoff-irrelevant strategies

We have a map $f^0$ with no fixed points other than $\sigma$.

- The map $f^0$ is meant to be a “better-reply” function.
Difficult Direction 3: add payoff-irrelevant strategies

We have a map $f^0$ with no fixed points other than $\sigma$.

- The map $f^0$ is meant to be a “better-reply” function.
- But $f^0_n$ is defined over $\Sigma$ and not just $\Sigma_{-n}$. 
Difficult Direction 3: add payoff-irrelevant strategies

We have a map $f^0$ with no fixed points other than $\sigma$. 

- The map $f^0$ is meant to be a “better-reply” function.
- But $f_n^0$ is defined over $\Sigma$ and not just $\Sigma_{-n}$.
- Construct an equivalent game $\tilde{G}$ that incorporates this.
Difficult Direction 3: add payoff-irrelevant strategies

We have a map $f^0$ with no fixed points other than $\sigma$.

- The map $f^0$ is meant to be a "better-reply" function.
- But $f_n^0$ is defined over $\Sigma$ and not just $\Sigma_{-n}$.
- Construct an equivalent game $\tilde{G}$ that incorporates this.
- Strategy set of player $n$ is $\tilde{S}_n \equiv S_n \times S_{n+1}$, where the 2nd-coordinate is payoff-irrelevant.
Difficult Direction 3: add payoff-irrelevant strategies

We have a map $f^0$ with no fixed points other than $\sigma$.

- The map $f^0$ is meant to be a “better-reply” function.
- But $f^0_n$ is defined over $\Sigma$ and not just $\Sigma_{-n}$.
- Construct an equivalent game $\tilde{G}$ that incorporates this.
- Strategy set of player $n$ is $\tilde{S}_n \equiv S_n \times S_{n+1}$, where the 2nd-coordinate is payoff-irrelevant.
- Construct a map $\tilde{f}^0_n : \tilde{\Sigma}_{-n} \to \tilde{\Sigma}_n$ which uses player $n - 1$’s 2nd-coordinate choice to determine nth component in $f^0$. 
The proof needs a Delaunay triangulation of $\tilde{\Sigma}_n$

**Figure:** Horizontal axis represents the duplicated strategy set in the equivalent game. Vertical axis represents the original strategy set.
It is NOT a Delaunay triangulation
It is NOT a Delaunay triangulation
Delaunay triangulation for points in general position

Let $C = co\{ x_0, x_1, \ldots, x_k \}$ in $\mathbb{R}^d$ be $d$-dimensional.
Delaunay triangulation for points in general position

- Let $C = \text{co}\{x_0, x_1, \ldots, x_k\}$ in $\mathbb{R}^d$ be $d$-dimensional.
- Suppose $x_i$’s are in general position for spheres in $\mathbb{R}^d$. 

Govindan, Laraki, & Pahl
Delaunay triangulation for points in general position

- Let $C = \text{co}\{x_0, x_1, \ldots, x_k\}$ in $\mathbb{R}^d$ be $d$-dimensional.
- Suppose $x_i$’s are in general position for spheres in $\mathbb{R}^d$.
- The Delaunay triangulation of $C$ is constructed as follows.
Delaunay triangulation for points in general position

- Let $C = \text{co}\{ x_0, x_1, \ldots, x_k \}$ in $\mathbb{R}^d$ be $d$-dimensional.
- Suppose $x_i$’s are in general position for spheres in $\mathbb{R}^d$.
- The Delaunay triangulation of $C$ is constructed as follows.
- Let $D = \text{co}\{ (x_i, \|x_i\|^2) \in \mathbb{R}^{d+1} \text{, such that } i = 0, 1, \ldots k \}$. 

Govindan, Laraki, & Pahl
Delaunay triangulation for points in general position

Let $C = \text{co}\{ x_0, x_1, \ldots, x_k \}$ in $\mathbb{R}^d$ be $d$-dimensional.

Suppose $x_i$’s are in general position for spheres in $\mathbb{R}^d$.

The Delaunay triangulation of $C$ is constructed as follows.

Let $D = \text{co}\{ (x_i, \|x_i\|^2) \in \mathbb{R}^{d+1}, \text{ such that } i = 0, 1, \ldots k \}$.

Let $D_0$ be the lower convex envelope of $D$. 
Delaunay triangulation for points in general position

- Let $C = \text{co}\{x_0, x_1, \ldots, x_k\}$ in $\mathbb{R}^d$ be $d$-dimensional.
- Suppose $x_i$’s are in **general position** for spheres in $\mathbb{R}^d$.
- The **Delaunay triangulation** of $C$ is constructed as follows.
- Let $D = \text{co}\{(x_i, \|x_i\|^2) \in \mathbb{R}^{d+1}, \text{ such that } i = 0, 1, \ldots k\}$.
- Let $D_0$ be the lower convex envelope of $D$.
- $D_0$ is the graph of a piecewise linear convex function $\rho : C \rightarrow \mathbb{R}$
Delaunay triangulation for points in general position

- Let $C = \text{co}\{x_0, x_1, \ldots, x_k\}$ in $\mathbb{R}^d$ be $d$-dimensional.
- Suppose $x_i$'s are in general position for spheres in $\mathbb{R}^d$.
- The Delaunay triangulation of $C$ is constructed as follows.
- Let $D = \text{co}\{(x_i, \|x_i\|^2) \in \mathbb{R}^{d+1}, \text{ such that } i = 0, 1, \ldots k\}$.
- Let $D_0$ be the lower convex envelope of $D$.
- $D_0$ is the graph of a piecewise linear convex function
  \[\rho : C \rightarrow \mathbb{R}\]
- Simplices where $\rho$ is linear forms a Delaunay triangulation.
VERY Difficult Direction 5: Add Bonus

- Consider a sufficiently fine Delaunay triangulation of $\tilde{\Sigma}_n$. 
VERY Difficult Direction 5: Add Bonus

- Consider a sufficiently fine Delaunay triangulation of $\tilde{\Sigma}_n$.
- Add vertices of the triangulation as pure strategies to $G$. 

Govindan, Laraki, & Pahl
Consider a sufficiently fine Delaunay triangulation of $\tilde{\Sigma}_n$. Add vertices of the triangulation as pure strategies to $G$. Give a bonus to the new strategies using $\tilde{f}^0$ (following several tricks, as in Govindan & Wilson 2005).
VERY Difficult Direction 5: Add Bonus

- Consider a sufficiently fine Delaunay triangulation of $\tilde{\Sigma}_n$.
- **Add** vertices of the triangulation as pure strategies to $G$.
- **Give a bonus** to the new strategies using $\tilde{f}^0$ (following several tricks, as in Govindan & Wilson 2005)
- The $\rho$ in Delaunay triangulation will be part of the bonus.
VERY Difficult Direction 5: Add Bonus

- Consider a sufficiently fine Delaunay triangulation of $\tilde{\Sigma}_n$.
- **Add** vertices of the triangulation as pure strategies to $G$.
- **Give a bonus** to the new strategies using $\tilde{f}^0$ (following several tricks, as in Govindan & Wilson 2005)
- The $\rho$ in Delaunay triangulation will be part of the bonus.
- Conclude using that $f^0$ has only one fixed point $\sigma$. 
Importance of not adding/removing best replies in IIA

Consider the following three-player game $G$, where player 3 has a unique action (is a dummy player).

$$
G = \begin{array}{ccc}
& l & r \\
t & (6, 6, 1) & (0, 0, 1) \\
b & (0, 0, 1) & (6, 6, 1)
\end{array}
$$
Importance of not adding/removing best replies in IIA

Consider the following three-player game $G$, where player 3 has a unique action (is a dummy player).

$$G = \begin{array}{c|cc}
 & l & r \\
\hline
 t & (6, 6, 1) & (0, 0, 1) \\
 b & (0, 0, 1) & (6, 6, 1) \\
\end{array}$$

Three Nash equilibria

- Two strict, $(t, l)$ and $(b, r)$ (index +1)
Importance of not adding/removing best replies in IIA

Consider the following three-player game $G$, where player 3 has a unique action (is a dummy player).

\[
G = \begin{array}{c|cc}
& l & r \\
\hline
t & (6, 6, 1) & (0, 0, 1) \\
b & (0, 0, 1) & (6, 6, 1) \\
\end{array}
\]

Three Nash equilibria

- Two strict, $(t, l)$ and $(b, r)$ (index +1)
- One completely mixed $\sigma$, with index −1.
Importance of not adding/removing best replies in IIA

Consider the following three-player game $G$, where player 3 has a unique action (is a dummy player).

$$
G = \begin{pmatrix}
  l & r \\
  t & (6, 6, 1) & (0, 0, 1) \\
  b & (0, 0, 1) & (6, 6, 1)
\end{pmatrix}
$$

Three Nash equilibria

- Two strict, $(t, l)$ and $(b, r)$ (index +1)
- One completely mixed $\sigma$, with index $-1$.
- Adding strategies leads to larger game where $\sigma$ is the unique equilibrium.
Importance of not adding/removing best replies in IIA

Consider the following three-player game $G$, where player 3 has a unique action (is a dummy player).

$G = \begin{array}{ccc}
    & l & r \\
    t & (6,6,1) & (0,0,1) \\
    b & (0,0,1) & (6,6,1) \\
\end{array}$

Three Nash equilibria

- Two strict, $(t,l)$ and $(b,r)$ (index +1)
- One completely mixed $\sigma$, with index $-1$.
- Adding strategies leads to larger game where $\sigma$ is the unique equilibrium.
- This cannot be done for a two player game!

Govindan, Laraki, & Pahl
−1 equilibrium can become unique by new strategies!

![Game Matrix](image)

Govindan, Laraki, & Pahl
1 equilibrium can become unique by new strategies!

\[
\begin{array}{ccc|ccc}
 & L & & & R \\
 T & t & (6, 6, 1) & r & (0, 0, 1) & (3, 3, 0) \\
 & b & (0, 0, 1) & (6, 6, 1) & \\
 B & (3, 0, 1) & & (0, 3, 1) & \\
\end{array}
\]

Something wrong?

\[
\begin{array}{ccc|ccc}
 & L & & & R \\
 T & t & (−3, 0, 4) & (1, 4, 0) & (1, 0, 1) \\
 & b & (1, 4, 0) & (1, 4, 0) & \\
 B & (3, 0, 0) & & (0, 3, 0) & \\
\end{array}
\]

\[
\begin{array}{ccc|ccc}
 & L & & & R \\
 T & t & (1, 4, 0) & (1, 4, 0) & (3, 0, 1) \\
 & b & (1, 4, 0) & (−3, 0, 4) & \\
 B & (3, 0, 0) & & (0, 3, 0) & \\
\end{array}
\]

Govindan, Laraki, & Pahl
−1 equilibrium can become unique by new strategies!

\[\begin{array}{c|cc|c}
& L & R \\
\hline
T & & \\
& t & (6, 6, 1) & (0, 0, 1) \\
& b & (0, 0, 1) & (6, 6, 1) \\
& (3, 0, 1) & (0, 3, 1) \\
B & & \\
\end{array}\]

Something wrong?

**NO**: \((G, \sigma)\) is not equivalent to \((\hat{G}, \sigma)\) because some of strategies we added are best replies to the equilibrium!
−1 equilibrium can become unique by new strategies!

Something wrong?

NO: \((G, \sigma)\) is not equivalent to \((\hat{G}, \sigma)\) because some of strategies we added are best replies to the equilibrium!

Open problem: Can any equilibrium be made unique by adding (non necessarily inferior replies) strategies and players?
Extension to non generic games: a first attempt

Fact 1: A non isolated equilibrium $\sigma$ of a game $G$ cannot be made unique in a larger game $\hat{G}$ obtained from $G$ by adding inferior replies (our result is sharp).
Extension to non generic games: a first attempt

Fact 1: A non isolated equilibrium $\sigma$ of a game $G$ cannot be made unique in a larger game $\hat{G}$ obtained from $G$ by adding inferior replies (our result is sharp).

Proof: By contradiction, let $\sigma_n \to \sigma$ be a sequence of equilibria of $G$. Since $\sigma$ is the unique equilibrium of $\hat{G}$, there is a (up to a subsequence) a fixed player $i$ and a fixed profitable best reply deviation $\theta_i$ to $\sigma_n$ in $\hat{G}$. By continuity, $\theta_i$ is a best reply to $\sigma$ in $\hat{G}$: a contradiction.
Extension to non generic games: a first attempt

**Fact 1:** A non isolated equilibrium $\sigma$ of a game $G$ cannot be made unique in a larger game $\hat{G}$ obtained from $G$ by adding inferior replies (our result is sharp).

**Proof:** By contradiction, let $\sigma_n \to \sigma$ be a sequence of equilibria of $G$. Since $\sigma$ is the unique equilibrium of $\hat{G}$, there is a (up to a subsequence) a fixed player $i$ and a fixed profitable best reply deviation $\theta_i$ to $\sigma_n$ in $\hat{G}$. By continuity, $\theta_i$ is a best reply to $\sigma$ in $\hat{G}$: a contradiction.

**Fact 2:** A strict subset $U$ of an equilibrium component $C$ in $G$ can never be made the unique component in a larger game $\hat{G}$ obtained from $G$ by adding inferior replies.
Extension to non generic games: a first attempt

Fact 1: A non isolated equilibrium $\sigma$ of a game $G$ cannot be made unique in a larger game $\hat{G}$ obtained from $G$ by adding inferior replies (our result is sharp).

Proof: By contradiction, let $\sigma_n \rightarrow \sigma$ be a sequence of equilibria of $G$. Since $\sigma$ is the unique equilibrium of $\hat{G}$, there is a (up to a subsequence) a fixed player $i$ and a fixed profitable best reply deviation $\theta_i$ to $\sigma_n$ in $\hat{G}$. By continuity, $\theta_i$ is a best reply to $\sigma$ in $\hat{G}$: a contradiction.

Fact 2: A strict subset $U$ of an equilibrium component $C$ in $G$ can never be made the unique component in a larger game $\hat{G}$ obtained from $G$ by adding inferior replies.

Proof: By contradiction, $U$ must be closed and since $C$ is connected, there is $\sigma \in U$, and $\sigma_n \in C$ but not in $U$ such that $\sigma_n \rightarrow \sigma$, then we use same argument as above.
First attempt does not extend to all games!

- **Corollary:** Only an (entire) equilibrium component can be made unique by adding inferior replies.
First attempt does not extend to all games!

- **Corollary:** Only an (entire) equilibrium component can be made unique by adding inferior replies.

- **BIG PROBLEM:** There are games where no equilibrium component can be made unique by adding inferior replies.
First attempt does not extend to all games!

- **Corollary:** Only an (entire) equilibrium component can be made unique by adding inferior replies.

- **BIG PROBLEM:** There are games where no equilibrium component can be made unique by adding inferior replies.

**Proof:** In Hauk & Hurkens example, no equilibrium component has index +1.
First attempt does not extend to all games!

- **Corollary:** Only an (entire) equilibrium component can be made unique by adding inferior replies.

- **BIG PROBLEM:** There are games where no equilibrium component can be made unique by adding inferior replies.

**Proof:** In Hauk & Hurkens example, no equilibrium component has index +1.

- **An open problem:**
  
  Can a +1 index equilibrium component be made unique by adding inferior replies?
Second attempt: an axiomatic approach

We want to construct a correspondence $\Phi$ that selects for each finite game $G$, a set of subsets of Nash equilibria of $G$ (called sustainable sets for $G$) satisfying the following axioms:

- **A$_1$: Existence:** Every game has a sustainable set.
- **A$_2$: IIA:** If a set is sustainable for a game, it remains sustainable after adding or removing inferior replies.
- **A$_3$: Invariance:** Equivalent games (obtained by additive positive transformation of payoffs, or addition/deletion of duplicate of mixte strategies) have equivalent solutions.
- **A$_4$: Robustness:** If a set is sustainable for $G$, then any nearby game has a nearby sustainable set.
- **A$_5$: Minimality:** If $\Psi$ satisfies A$_1$ to A$_4$ then $\Phi \subset \Psi$.

Theorem $\Phi$ satisfies A$_1$ to A$_5$ iff it associates to each game: its set of Nash equilibrium components with positive index.
Second attempt: an axiomatic approach

We want to construct a correspondence $\Phi$ that selects for each finite game $G$, a set of subsets of Nash equilibria of $G$ (called sustainable sets for $G$) satisfying the following axioms:

- **A1: Existence:** Every game has a sustainable set.
Second attempt: an axiomatic approach

We want to construct a correspondence \( \Phi \) that selects for each finite game \( G \), a set of subsets of Nash equilibria of \( G \) (called sustainable sets for \( G \)) satisfying the following axioms:

- **A1: Existence:** Every game has a sustainable set.
- **A2: IIA:** If a set is sustainable for a game, it remains sustainable after adding or removing inferior replies.
Second attempt: an axiomatic approach

We want to construct a correspondence $\Phi$ that selects for each finite game $G$, a set of subsets of Nash equilibria of $G$ (called sustainable sets for $G$) satisfying the following axioms:

- **A1: Existence:** Every game has a sustainable set.
- **A2: IIA:** If a set is sustainable for a game, it remains sustainable after adding or removing inferior replies.
- **A3: Invariance:** Equivalent games (obtained by additive positive transformation of payoffs, or addition/deletion of duplicate of mixte strategies) have equivalent solutions.
Second attempt: an axiomatic approach

We want to construct a correspondence $\Phi$ that selects for each finite game $G$, a set of subsets of Nash equilibria of $G$ (called sustainable sets for $G$) satisfying the following axioms:

- **$A_1$: Existence:** Every game has a sustainable set.
- **$A_2$: IIA:** If a set is sustainable for a game, it remains sustainable after adding or removing inferior replies.
- **$A_3$: Invariance:** Equivalent games (obtained by additive positive transformation of payoffs, or addition/deletion of duplicate of mixte strategies) have equivalent solutions.
- **$A_4$: Robustness:** If a set is sustainable for $G$, then any nearby game has a nearby sustainable set.
Second attempt: an axiomatic approach

We want to construct a correspondence $\Phi$ that selects for each finite game $G$, a set of subsets of Nash equilibria of $G$ (called sustainable sets for $G$) satisfying the following axioms:

- **A1: Existence:** Every game has a sustainable set.
- **A2: IIA:** If a set is sustainable for a game, it remains sustainable after adding or removing inferior replies.
- **A3: Invariance:** Equivalent games (obtained by additive positive transformation of payoffs, or addition/deletion of duplicate of mixte strategies) have equivalent solutions.
- **A4: Robustness:** If a set is sustainable for $G$, then any nearby game has a nearby sustainable set.
- **A5: Minimality:** If $\Psi$ satisfies $A1$ to $A4$ then $\Phi \subset \Psi$. 

Govindan, Laraki, & Pahl
Second attempt: an axiomatic approach

We want to construct a correspondence $\Phi$ that selects for each finite game $G$, a set of subsets of Nash equilibria of $G$ (called sustainable sets for $G$) satisfying the following axioms:

- $A_1$: Existence: Every game has a sustainable set.
- $A_2$: IIA: If a set is sustainable for a game, it remains sustainable after adding or removing inferior replies.
- $A_3$: Invariance: Equivalent games (obtained by additive positive transformation of payoffs, or addition/deletion of duplicate of mixte strategies) have equivalent solutions.
- $A_4$: Robustness: If a set is sustainable for $G$, then any nearby game has a nearby sustainable set.
- $A_5$: Minimality: If $\Psi$ satisfies $A_1$ to $A_4$ then $\Phi \subseteq \Psi$.

Theorem $\Phi$ satisfies $A_1$ to $A_5$ iff it associates to each game:
Second attempt: an axiomatic approach

We want to construct a correspondence $\Phi$ that selects for each finite game $G$, a set of subsets of Nash equilibria of $G$ (called sustainable sets for $G$) satisfying the following axioms:

- **A1: Existence:** Every game has a sustainable set.
- **A2: IIA:** If a set is sustainable for a game, it remains sustainable after adding or removing inferior replies.
- **A3: Invariance:** Equivalent games (obtained by additive positive transformation of payoffs, or addition/deletion of duplicate of mixte strategies) have equivalent solutions.
- **A4: Robustness:** If a set is sustainable for $G$, then any nearby game has a nearby sustainable set.
- **A5: Minimality:** If $\Psi$ satisfies $A1$ to $A4$ then $\Phi \subset \Psi$.

**Theorem** $\Phi$ satisfies $A1$ to $A5$ iff it associates to each game: its set of Nash equilibrium components with positive index.
Robustness in Myerson’s Essai

It is hard to see how cultural expectations of the mixed equilibrium could be sustained, however. For example, if the outcome \((b_1, b_2)\) occurred 100 times in a row (as must eventually happen in a sufficiently long history of plays of the mixed equilibrium), then surely each player would increase his or her subjective probability of the other player choosing \(b_1\) or \(b_2\) at the next play of the game. But when this subjective probability is increased above the probability in the mixed equilibrium, then choosing \(b_2\) or \(b_1\) would become optimal for each player, and so the repetition of the \((b_1, b_2)\) outcome should continue. Thus any evidence that pushes the players slightly away from the mixed equilibrium can ultimately move them all the way to one or the other of the pure-strategy equilibria.
Govindan, Laraki, & Pahl