Fast Algorithms for Rank-1 Bimatrix Games

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joint with

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Bimatrix games (A, B)

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T	1, 1	0, −2
B	0, −1	1, 0

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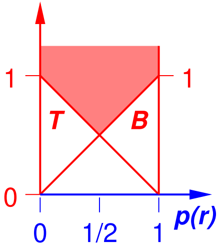

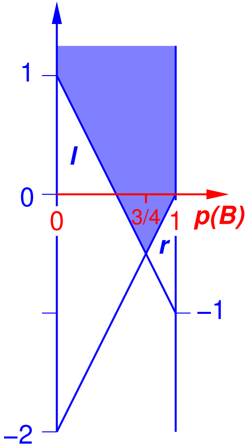

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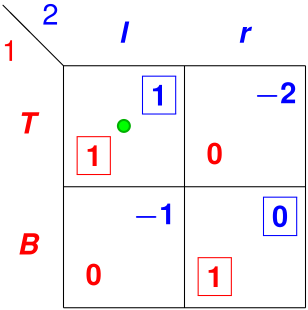

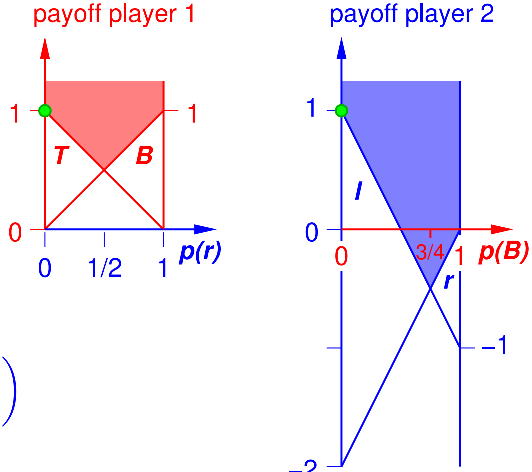

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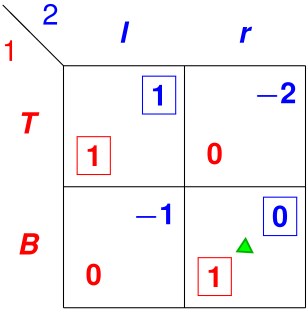

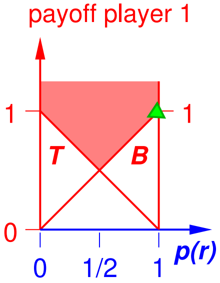

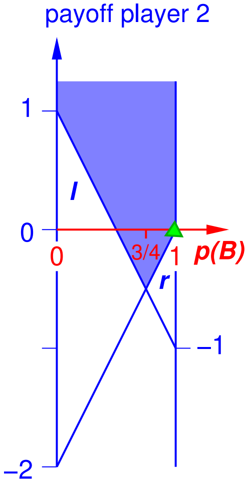

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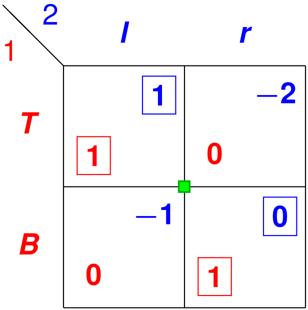
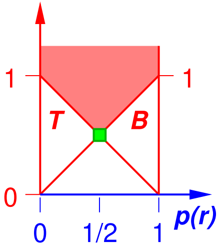
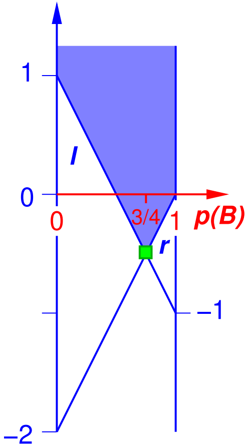


payoff player 1



payoff player 2



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Solving bimatrix games

The algorithmic problem

- Input: An $m \times n$ bimatrix game (A, B)
- Output: All its Nash equilibria, or just **one** equilibrium

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Approaches and challenges

- small games ($m, n < 30$): enumerate and match vertices of best-response polytopes defined by A and B
- deciding if more than one Nash equilibrium is NP-hard
- finding **one** Nash equilibrium: path-following, class PPAD, PPAD-complete (but may in practice work for $m, n \sim 1000$)

Nash equilibria of games with parameter λ

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0+\lambda \\ 0 & 1+\lambda \end{pmatrix}$$

$\lambda = \frac{1}{2}$:

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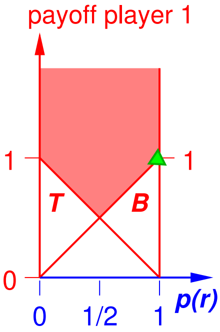

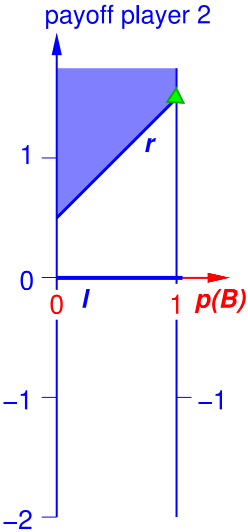

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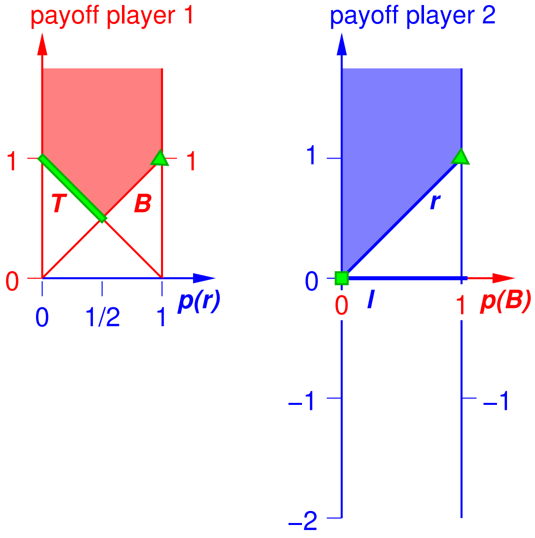

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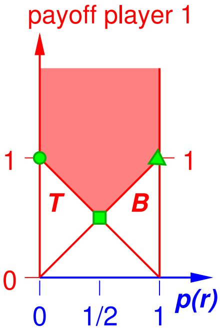

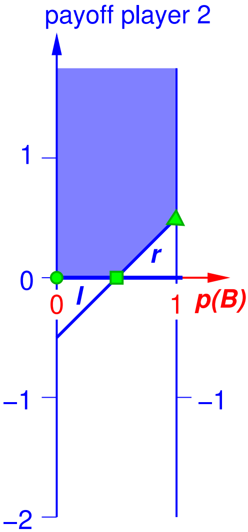

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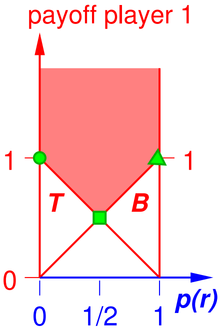

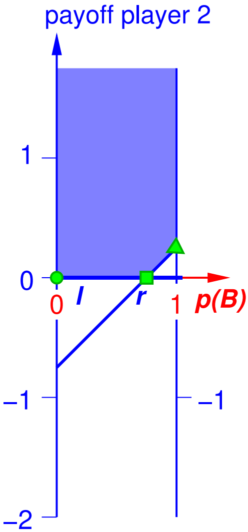

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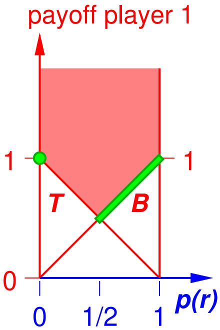

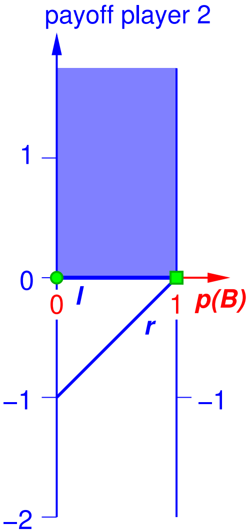

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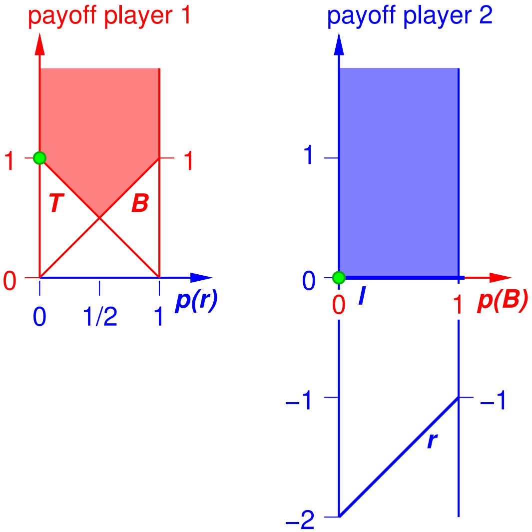

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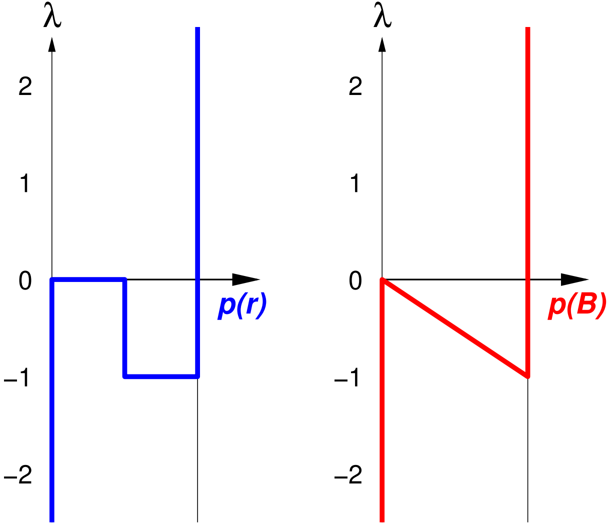

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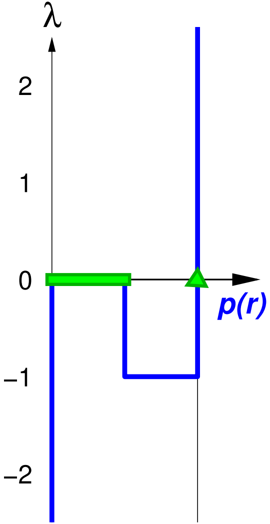

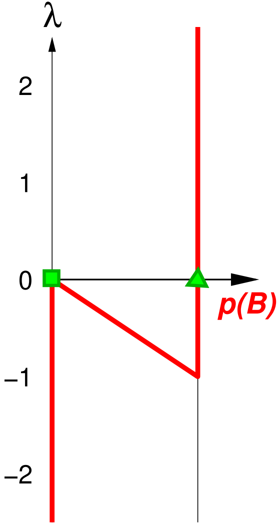

Global Newton Method [Govindan/Wilson 2003]

- $m \times n$ game (A, B), $c \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, $\quad \mathbb{1} = (1, 1, \ldots, 1)^\top$
- trace equilibria of $(A + c\lambda\mathbb{1}^\top, B + \mathbb{1}\lambda b^\top)$ for $\lambda \in \mathbb{R}$, that is, of

$$\left(A + \begin{pmatrix} c_1 & c_1 & \cdots & c_1 \\ c_2 & c_2 & \cdots & c_2 \\ \vdots & & & \\ c_m & c_m & \cdots & c_m \end{pmatrix} \lambda\,,\quad B + \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & & & \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} \lambda\right)$$

- b_j unique maximum of b $\Rightarrow$ for large λ the pure strategy j is dominant, with best response to j get Nash equilibrium (NE)
- follow this NE (using *complementary pivoting* = Lemke's algorithm for Linear Complementarity Problems) until $\lambda = 0$

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- = Lemke-Howson algorithm if $(c, b) = (0, \ldots, 0, 1, 0, \ldots, 0)$

Zero-sum games $(A, -A)$

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unique mixed equilibrium $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$

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in general:

- convex equilibrium set (generically: singleton)
- solution to a linear program (LP)

Rank-k games

rank of game **(A, B)** = rank($A + B$)

Rank-$\boldsymbol{k}$ games

rank of game $(\boldsymbol{A}, \boldsymbol{B})$ = rank($\boldsymbol{A} + \boldsymbol{B}$)

game of rank **0**: $(\boldsymbol{A}, -\boldsymbol{A})$ (zero-sum)

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An exact NE of a rank-**1** game can be found in polynomial time.

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rank$(A + B) = k$
$\Rightarrow \quad A + B = a^1 b^{1\top} + \cdots + a^k b^{k\top}$ for $a^i \in \mathbb{R}^m$, $b^i \in \mathbb{R}^n$

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$$\Rightarrow \quad (A, B) = (A, C + ab^\top) \text{ where } (A, C) \text{ has rank } k - 1$$

game of rank $\mathbf{1}$: $(A, B) = (A, -A + ab^\top)$

Rank reduction by intersection with hyperplane

The following are equivalent:

$$(\boldsymbol{x}, \boldsymbol{y}) \text{ NE of } (\boldsymbol{A}, \boldsymbol{C} + \boldsymbol{a}\boldsymbol{b}^\top)$$

$$\Leftrightarrow \quad (\boldsymbol{x}, \boldsymbol{y}) \text{ NE of } (\boldsymbol{A}, \boldsymbol{C} + \mathbb{1}\lambda\boldsymbol{b}^\top) \text{ and } \boldsymbol{x}^\top\boldsymbol{a} = \lambda$$

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Proof

Row payoffs: $\boldsymbol{A}$ unchanged, $\boldsymbol{x}$ stays best response to $\boldsymbol{y}$

Column payoffs: If $\boldsymbol{x}^\top\boldsymbol{a}=\lambda$, then

$$\boldsymbol{x}^\top(\boldsymbol{C}+\boldsymbol{a}\boldsymbol{b}^\top) = \boldsymbol{x}^\top\boldsymbol{C}+\lambda\boldsymbol{b}^\top = \boldsymbol{x}^\top\boldsymbol{C}+\boldsymbol{x}^\top\mathbb{1}\lambda\boldsymbol{b}^\top$$

$$= \quad \boldsymbol{x}^\top(\boldsymbol{C}+\mathbb{1}\lambda\boldsymbol{b}^\top)$$

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$\Leftrightarrow$ $(\boldsymbol{x}, \boldsymbol{y})$ NE of $(\boldsymbol{A} - \mathbb{1}\lambda\boldsymbol{b}^\top, \boldsymbol{C} + \mathbb{1}\lambda\boldsymbol{b}^\top)$ and $\boldsymbol{x}^\top\boldsymbol{a} = \lambda$

... where $(\boldsymbol{A} - \mathbb{1}\lambda\boldsymbol{b}^\top, \boldsymbol{C} + \mathbb{1}\lambda\boldsymbol{b}^\top)$ has rank $\boldsymbol{k} - \mathbf{1}$!

Example of rank-1 game

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$$

$$= -\boldsymbol{A} + \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$$

$$= -\boldsymbol{A} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} (1 \quad -1)$$

Example of rank-$\mathbf{1}$ game

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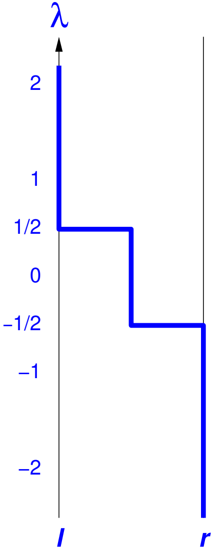
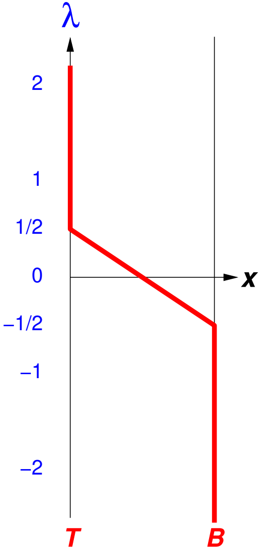
Now consider

$$= -A + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \lambda \, (1 \quad -1)$$





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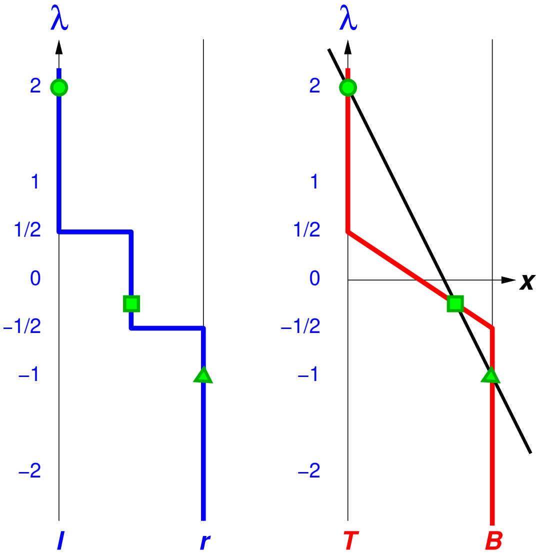
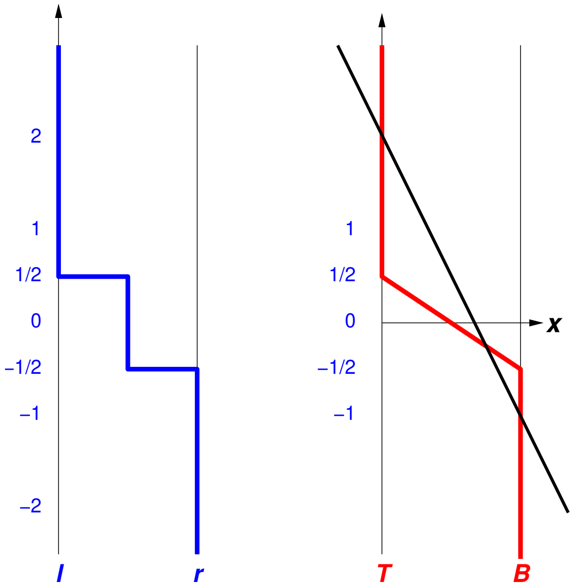
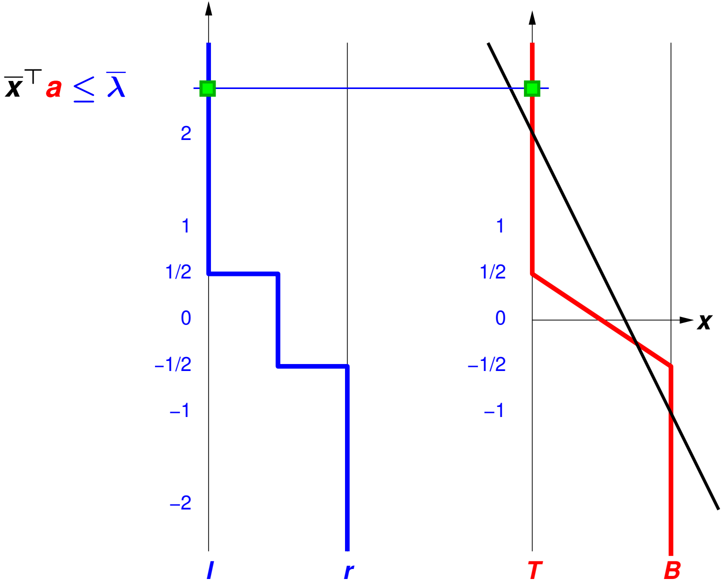
$x^\top a = x^\top \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \lambda$



Binary search for λ



Binary search for $\boldsymbol{\lambda}$



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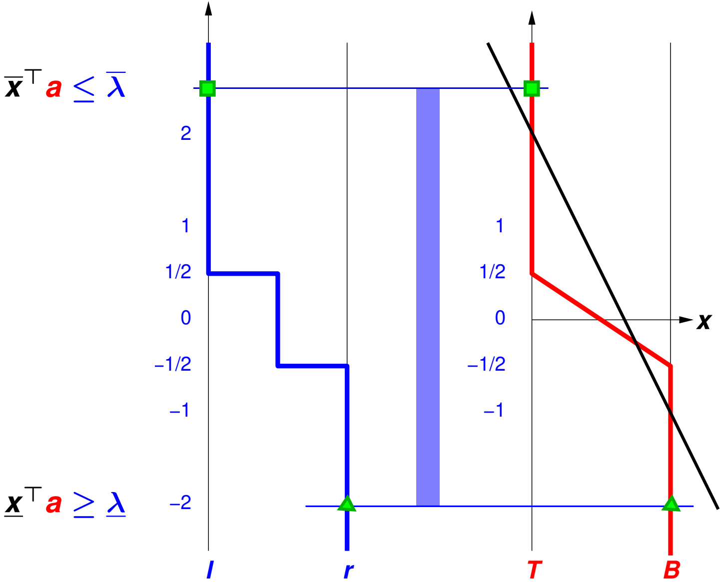

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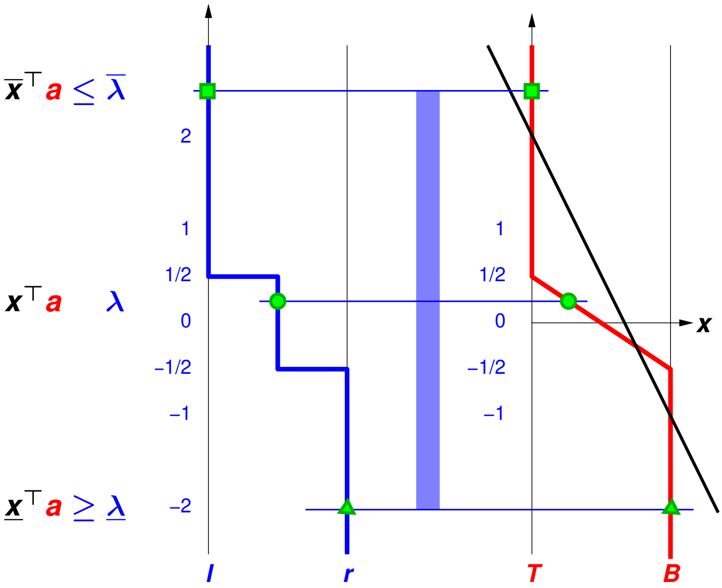

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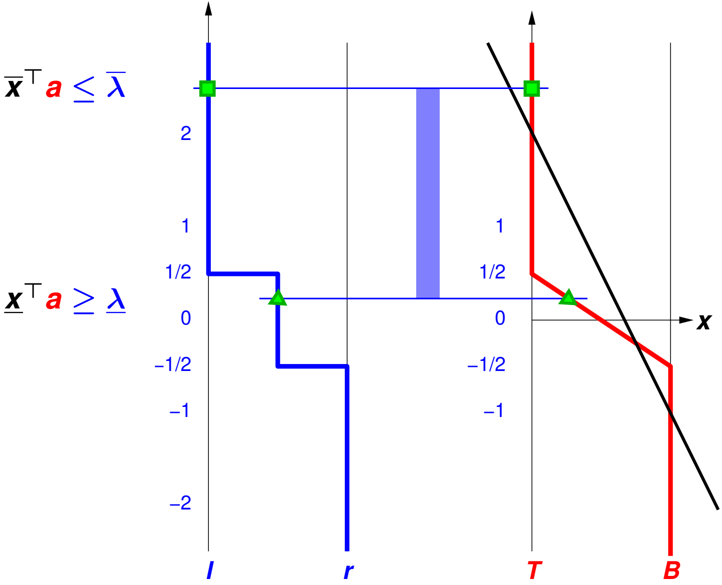

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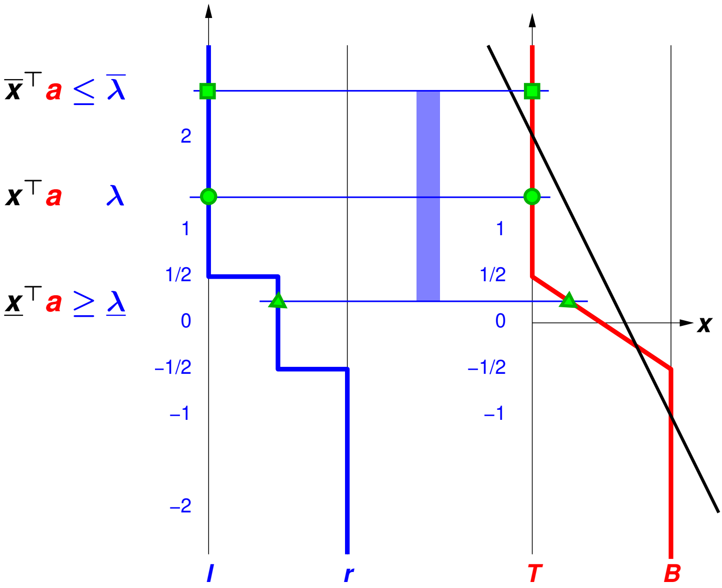

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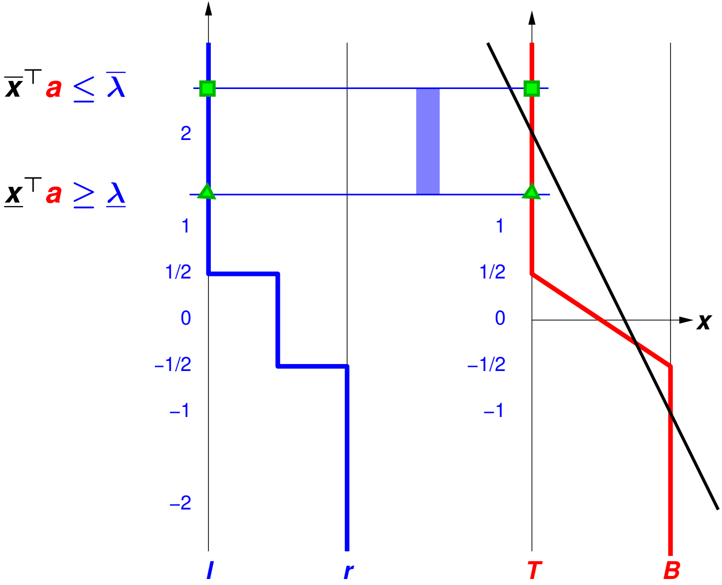


Binary search for λ

$\overline{x}^\top a \leq \overline{\lambda}$

$x^\top a = \lambda$

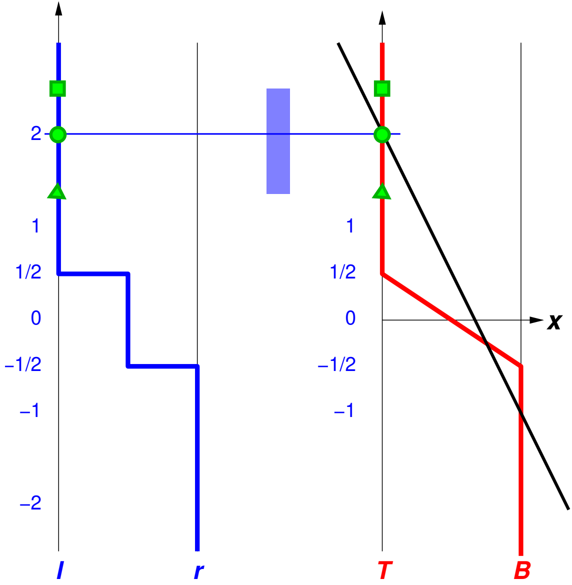
$\underline{x}^\top a \geq \underline{\lambda}$

$(\overline{x}, \overline{y})$ and $(\underline{x}, \underline{y})$

on same

path segment:

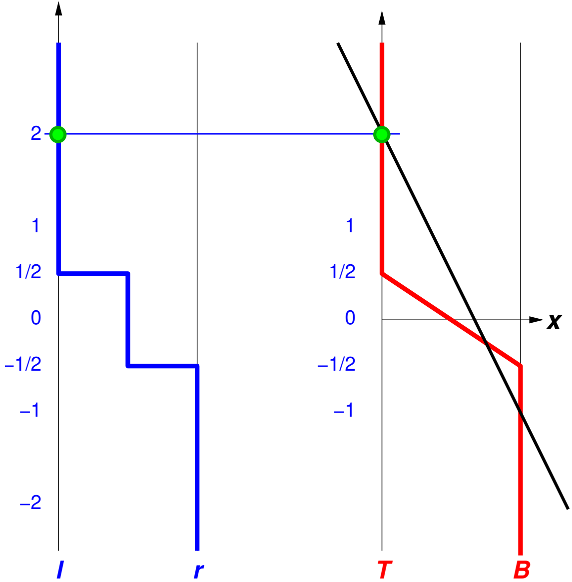
solve $x^\top a = \lambda$



Binary search for λ

$x^\top a = \lambda$

$(\overline{x}, \overline{y})$ and $(\underline{x}, \underline{y})$
on same
path segment:
solve $x^\top a = \lambda$



Algorithm: binary search for $\boldsymbol{\lambda}$

want: $\boldsymbol{\lambda}$ with NE $(\boldsymbol{x}, \boldsymbol{y})$ of $(\boldsymbol{A}, \boldsymbol{C} + \mathbb{1}\lambda\boldsymbol{b}^\top)$ and $\boldsymbol{x}^\top\boldsymbol{a} = \lambda$

$(\Rightarrow \boldsymbol{\lambda} \in \mathrm{conv}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\})$

initialize: $\overline{\lambda} = \max\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} + 1, \quad \underline{\lambda} = \min\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} - 1$

---

repeat $\quad \lambda \leftarrow (\overline{\lambda} - \underline{\lambda})/2$

$(\boldsymbol{x}, \boldsymbol{y}) \leftarrow$ NE of $(\boldsymbol{A}, \boldsymbol{C} + \mathbb{1}\lambda\boldsymbol{b}^\top)$

if $\lambda \leq \boldsymbol{x}^\top\boldsymbol{a}$: $\quad (\underline{\lambda}, \underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) \leftarrow (\lambda, \boldsymbol{x}, \boldsymbol{y})$

else $\quad (\overline{\lambda}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}) \leftarrow (\lambda, \boldsymbol{x}, \boldsymbol{y})$

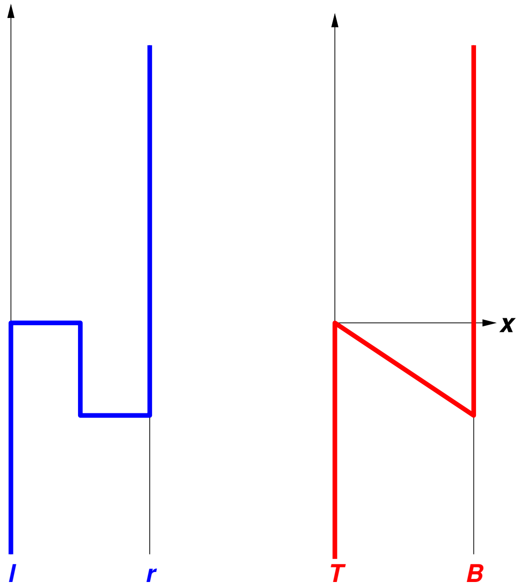
invariant : $\underline{\lambda} < \overline{\lambda}$, $\quad \boxed{\underline{\lambda} \leq \underline{\boldsymbol{x}}^\top\boldsymbol{a}, \quad \overline{\boldsymbol{x}}^\top\boldsymbol{a} \leq \overline{\lambda}}$

until $\quad (\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}})$ and $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}})$ are on same path segment

---

finish: solve for $\boldsymbol{x}^\top\boldsymbol{a} = \lambda$ on that segment.

Does this work for $C \neq -A$ (higher rank)?



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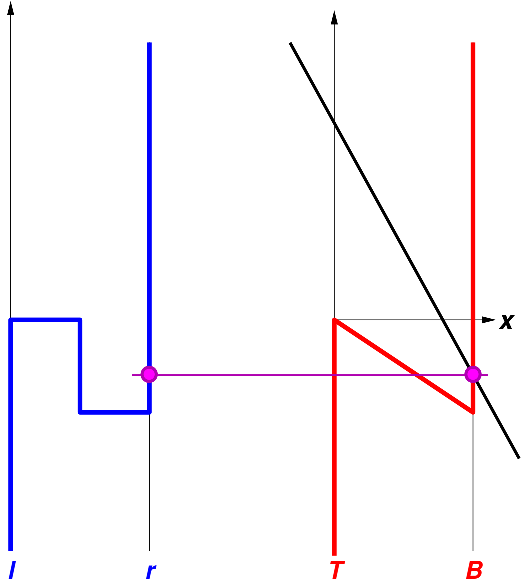
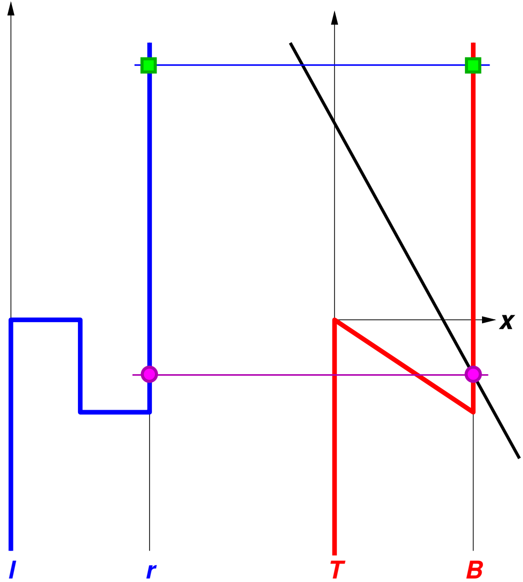
one NE



Does this work for $\boldsymbol{C} \neq -\boldsymbol{A}$ (higher rank)?

$\overline{\boldsymbol{x}}^\top \boldsymbol{a} \leq \overline{\lambda}$

one NE



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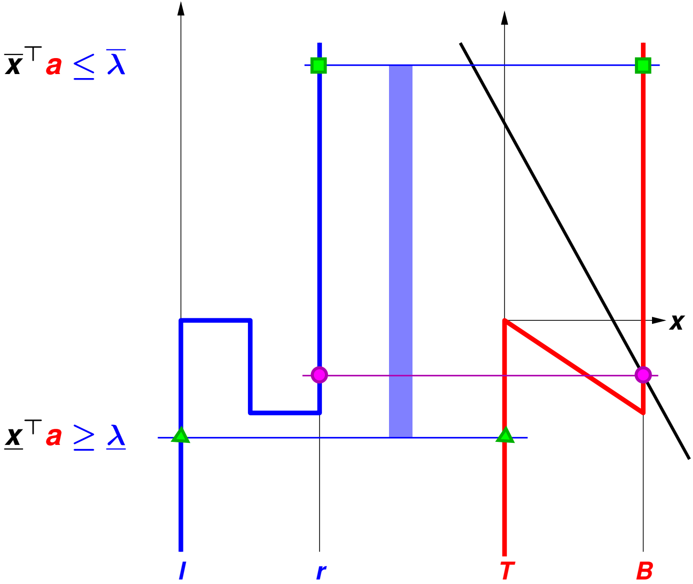

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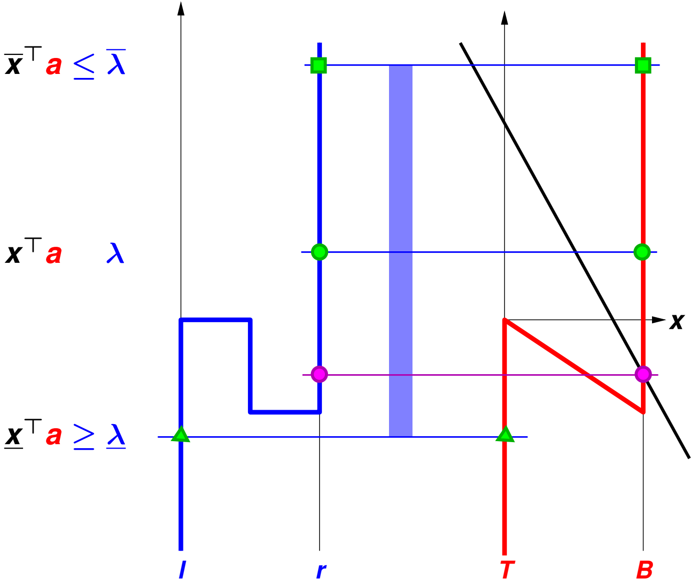

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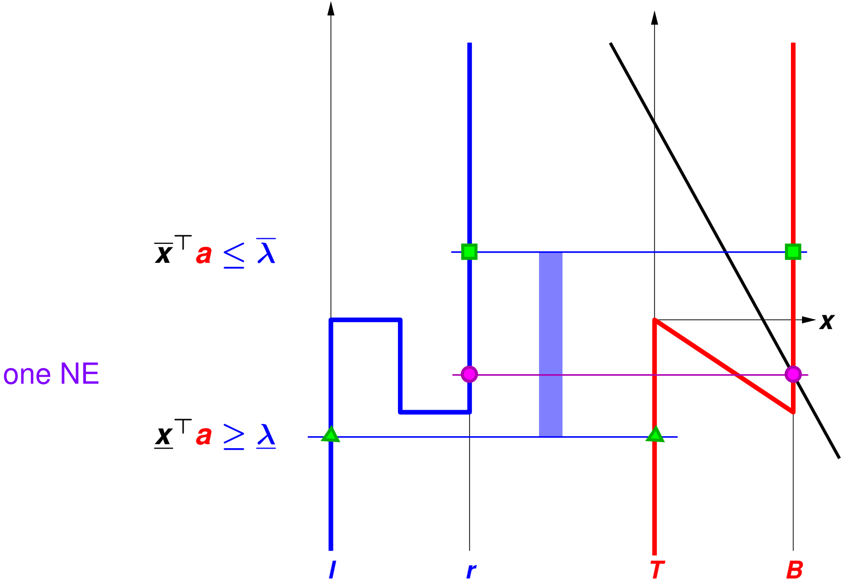

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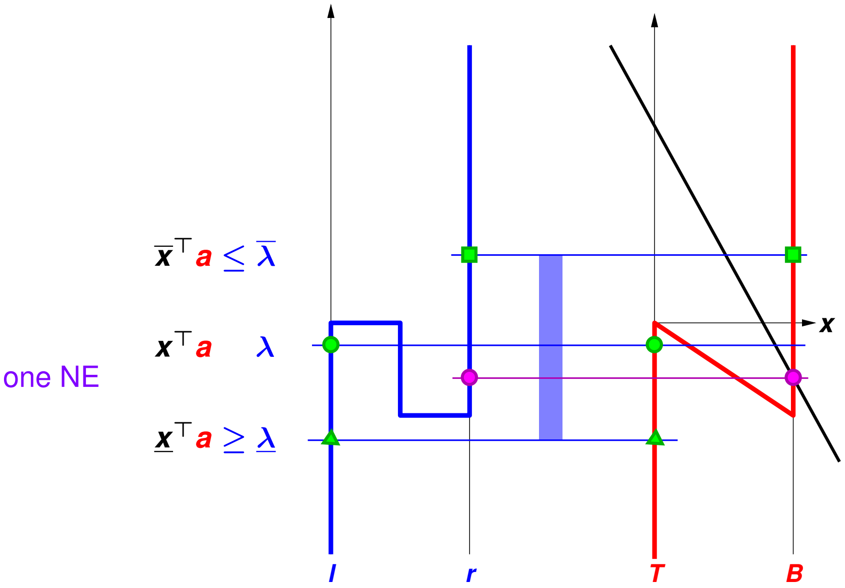

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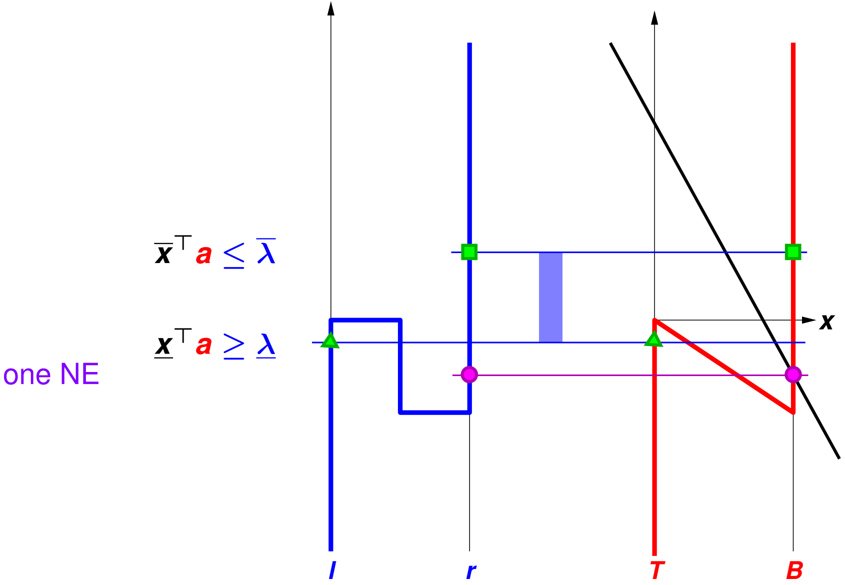

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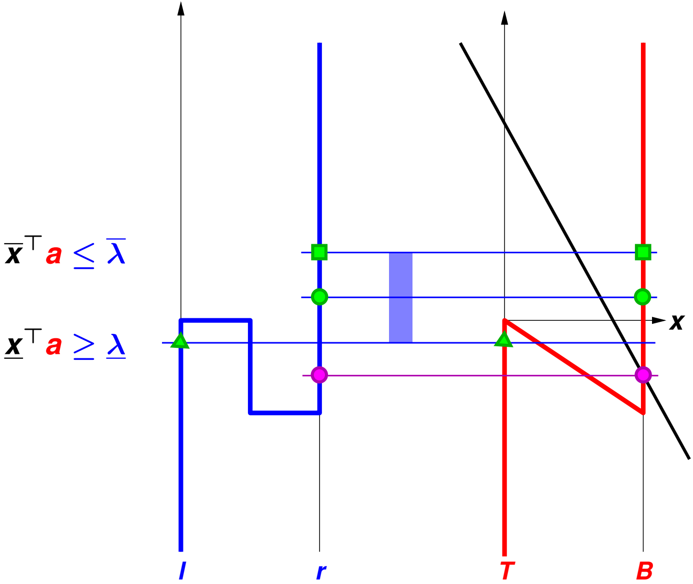


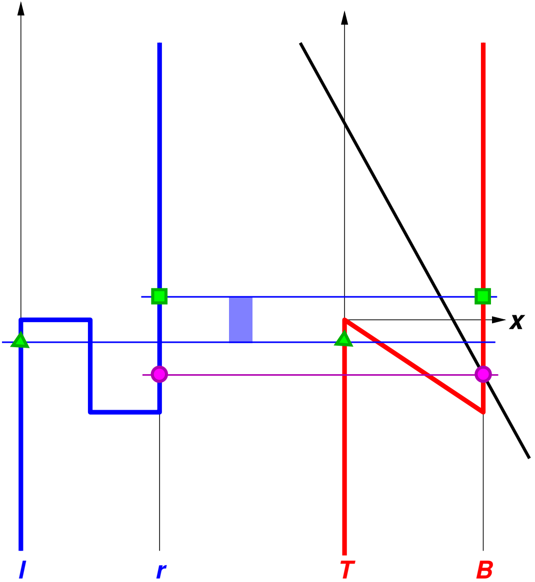
one NE

Does this work for $C \neq -A$ (higher rank)? **No!**

$\overline{x}^\top a \leq \overline{\lambda}$

$\underline{x}^\top a \geq \underline{\lambda}$

one NE



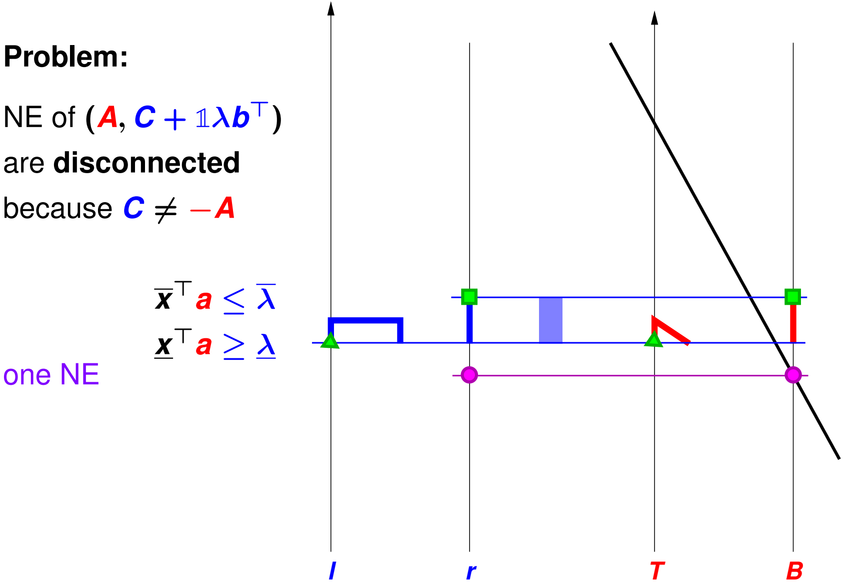
Does this work for $C \neq -A$ (higher rank)? **No!**

Problem:

NE of $(A, C + \mathbb{1}\lambda b^\top)$

are **disconnected**

because $C \neq -A$

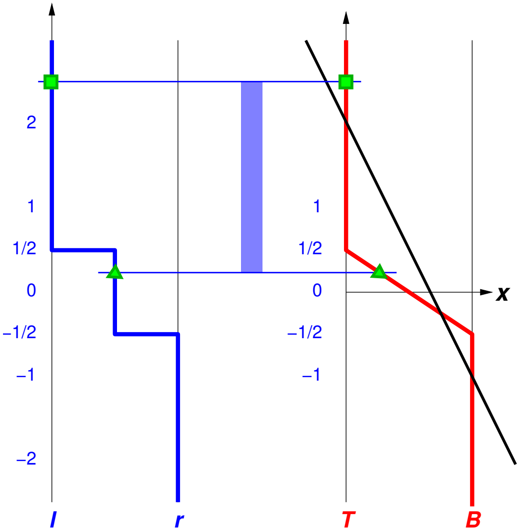


Monotone paths

The game
$(A, -A + \mathbb{1}\lambda b^\top)$
has the same
convex NE set
as the zero-sum game

$(A - \mathbb{1}\lambda b^\top, -A + \mathbb{1}\lambda b^\top)$

$\Rightarrow$ the λ-path
is monotone



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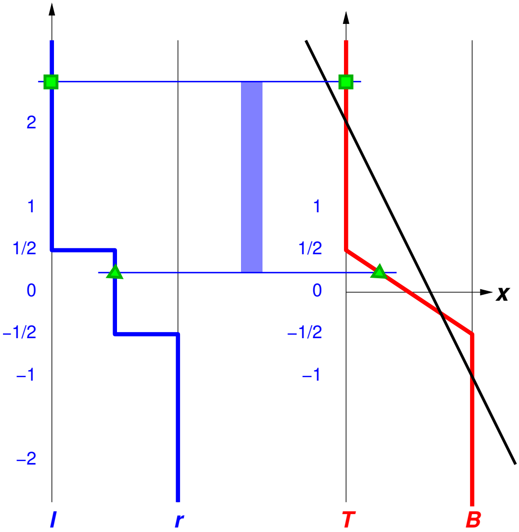

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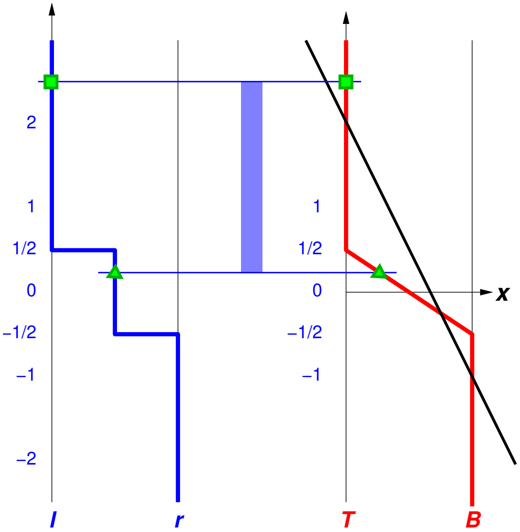
in polynomial time



It would have been too good to be true

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... if binary search to find one NE of a rank-***k*** game worked (recursively) with the general rank reduction.

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Rank-**3** bimatrix games are PPAD-hard.

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Theorem [Mehta, *STOC* 2014]

Rank-**3** bimatrix games are PPAD-hard.

Announcement [Chen, Orfanou, Paparas, Yannakakis 2016]

Rank-**2** bimatrix games are PPAD-hard.

Rank-$\mathbf{1}$ game

$(\boldsymbol{x}, \boldsymbol{y})$ NE of $(\boldsymbol{A}, -\boldsymbol{A} + \boldsymbol{a}\boldsymbol{b}^\top)$

$\Leftrightarrow$ $(\boldsymbol{x}, \boldsymbol{y})$ NE of $(\boldsymbol{A}, -\boldsymbol{A} + \mathbb{1}\boldsymbol{\lambda}\boldsymbol{b}^\top)$ and $\boldsymbol{x}^\top \boldsymbol{a} = \boldsymbol{\lambda}$

$\Leftrightarrow$ $(\boldsymbol{x}, \boldsymbol{y})$ NE of $(\boldsymbol{A} - \mathbb{1}\boldsymbol{\lambda}\boldsymbol{b}^\top, -\boldsymbol{A} + \mathbb{1}\boldsymbol{\lambda}\boldsymbol{b}^\top)$ and $\boldsymbol{x}^\top \boldsymbol{a} = \boldsymbol{\lambda}$

parameterized zero-sum game,

parameterized LP

LP for zero-sum game

NE $(\boldsymbol{x}, \boldsymbol{y})$ of zero-sum game $(\boldsymbol{A}, -\boldsymbol{A})$

$\Leftrightarrow$ $\boldsymbol{y}$ minmax strategy of column player

$\Leftrightarrow$ $\boldsymbol{y}, \boldsymbol{u}$ solve LP

$$\text{minimize } \boldsymbol{u} \quad \text{subject to} \quad \begin{aligned} \boldsymbol{A}\boldsymbol{y} &\leq \mathbb{1}\boldsymbol{u} \\ \mathbb{1}^\top \boldsymbol{y} &= \mathbf{1} \\ \boldsymbol{y} &\geq \mathbf{0} \end{aligned}$$

dual solution $\boldsymbol{x}$: maxmin strategy of row player

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---

write LP as

$$\text{minimize } \boldsymbol{u} \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{y} \leq \mathbb{1}\boldsymbol{u}, \quad \boldsymbol{y} \in \boldsymbol{Y}$$

LP for parameterized zero-sum game

NE $(\boldsymbol{x}, \boldsymbol{y})$ of zero-sum game $(\boldsymbol{A} - \mathbb{1}\lambda \boldsymbol{b}^\top, -\boldsymbol{A} + \mathbb{1}\lambda \boldsymbol{b}^\top)$

$\Leftrightarrow$ $\boldsymbol{y}, \boldsymbol{u}$ solve LP

minimize $\boldsymbol{u}$ subject to $(\boldsymbol{A} - \mathbb{1}\lambda \boldsymbol{b}^\top)\boldsymbol{y} \le \mathbb{1}\boldsymbol{u}, \quad \boldsymbol{y} \in \boldsymbol{Y}$

LP for parameterized zero-sum game

NE (x, y) of zero-sum game $(A - \mathbb{1}\lambda b^\top, -A + \mathbb{1}\lambda b^\top)$

$\Leftrightarrow$ y, u solve LP

minimize u subject to $(A - \mathbb{1}\lambda b^\top) y \le \mathbb{1} u, \quad y \in Y$

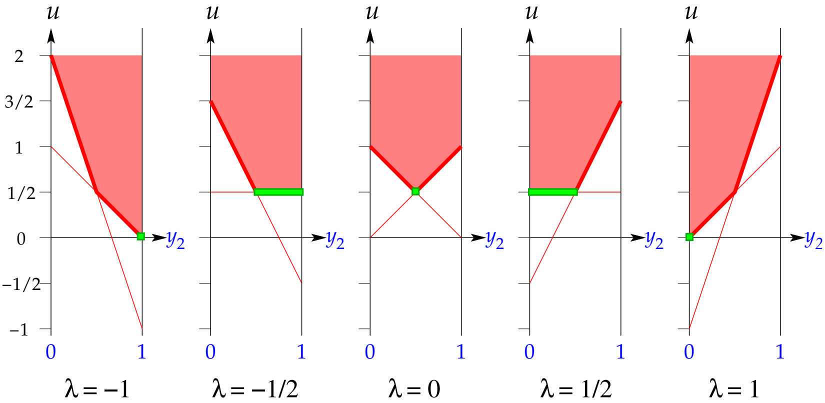
$\Leftrightarrow$ y, t solve LP

minimize $t - \lambda b^\top y \quad (= u)$

subject to $Ay \le \mathbb{1} t, \quad y \in Y$

Parameterized matrix (column bonuses)

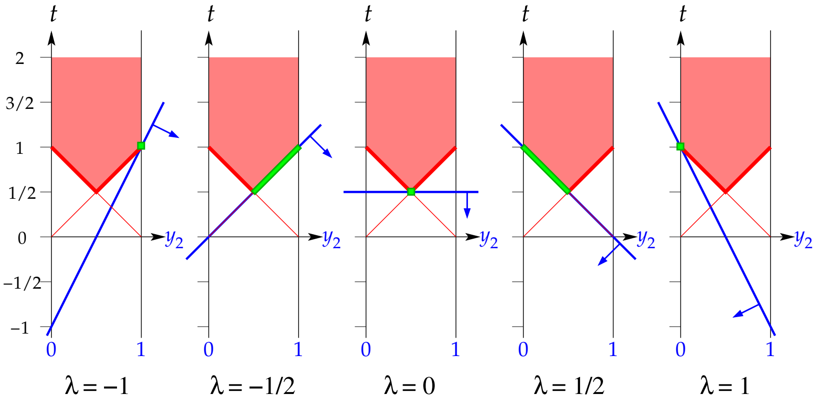
minimize $\boldsymbol{u}$ subject to $(\boldsymbol{A} - \mathbb{1}\lambda\boldsymbol{b}^\top)\boldsymbol{y} \le \mathbb{1}\boldsymbol{u}$, $\boldsymbol{y} \in \boldsymbol{Y}$



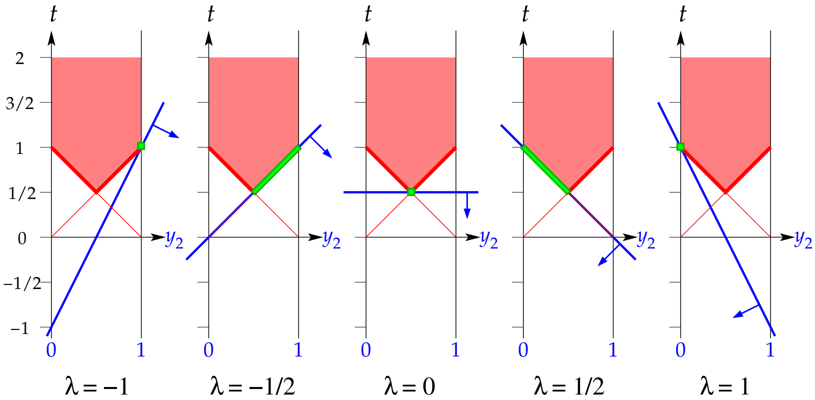
Equivalent: Parameterized objective function

minimize $\boldsymbol{t} - \lambda \boldsymbol{b}^\top \boldsymbol{y} \quad (= \boldsymbol{u})$

subject to $\quad \boldsymbol{A}\boldsymbol{y} \le \mathbb{1}\boldsymbol{t}, \quad \boldsymbol{y} \in \boldsymbol{Y}$



“Wrap-around” hyperplane defines the path



Hyperplane with normal vector $\boldsymbol{t} - \lambda \boldsymbol{b}^\top$ “wrapping around” upper-envelope polyhedron defines path-connected set

Algorithm 1: Binary search to find one NE

Input: Rank-**1** game $(\boldsymbol{A}, -\boldsymbol{A} + \boldsymbol{a}\boldsymbol{b}^\top)$. Consider

minimize $\boldsymbol{t} - \lambda \boldsymbol{b}^\top \boldsymbol{y}$

subject to $\boldsymbol{A}\boldsymbol{y} \le \mathbb{1}\boldsymbol{t}, \quad \boldsymbol{y} \in \boldsymbol{Y}$

Parameterized by λ, the path (generically), or sequence of faces (general case) on the polyhedron $\{(\boldsymbol{y}, \boldsymbol{t}) \mid \boldsymbol{A}\boldsymbol{y} \le \mathbb{1}\boldsymbol{t}\}$ defines strategies $\boldsymbol{y}$ for the zero-sum game with row payoffs $\boldsymbol{A} - \lambda\mathbb{1}\boldsymbol{b}^\top$, and dual solutions $\boldsymbol{x}$.

Want intersection with hyperplane $\{(\boldsymbol{x}, \lambda) \mid \boldsymbol{x}^\top \boldsymbol{a} = \lambda\}$.

- binary search on λ via conditions $\boldsymbol{x}^\top \boldsymbol{a} \le \lambda$ or $\boldsymbol{x}^\top \boldsymbol{a} \ge \lambda$ finds $\boldsymbol{x}^\top \boldsymbol{a} = \lambda$ in polynomial time, and thus one NE $(\boldsymbol{x}, \boldsymbol{y})$.

Algorithm 2: Enumeration of all NE

- Following the path by pivoting on the parameterized LP with alternate constraints $\boldsymbol{x}^\top \boldsymbol{a} \leq \boldsymbol{\lambda}$ or $\boldsymbol{x}^\top \boldsymbol{a} \geq \boldsymbol{\lambda}$ finds all NE.

  (Non-generically: get faces rather than edges.)

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  (Non-generically: get faces rather than edges.)

$\Rightarrow$ For rank-**1** games, **path-following** gives **all** NE rather than just one NE.

Number of NE of a rank-**1** game

One NE of a rank-**1** game can be found in polynomial time.

Are these games "less complex"?

Do they have fewer NE (e.g., polynomially many – asked by [Kannan & Theobald]) than general bimatrix games?

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$\Leftrightarrow$

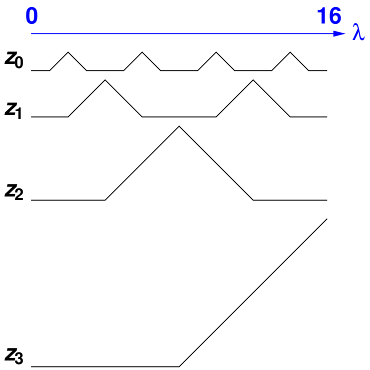
How many times can the path of solutions to a parameterized LP intersect a hyperplane?

Answer: Exponentially often!

Parameterized LP [Murty 1980]

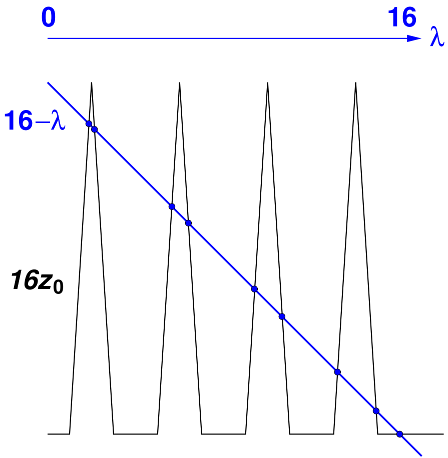
z_3	z_2	z_1	z_0	$\geq$	$0, 0, 0, 0$
1	0	0	0	$\geq$	$-8 + \lambda$
2	1	0	0	$\geq$	$-4 + \lambda$
2	2	1	0	$\geq$	$-2 + \lambda$
2	2	2	1	$\geq$	$-1 + \lambda$
64	16	4	1	$\rightarrow$	minimize

optimal $\boldsymbol{z}$:



Game with $2^{n-1} + 1$ equilibria

Hyperplane using only z_0 with $a^\top = (0, 0, 0, 16)$



Inspired by Murty

$n \times n$ rank-1 games with $2^n - 1$ many NE :

Let $p > 2$.

$$A = B^\top = \begin{pmatrix} p^0 & 2p^1 & 2p^2 & \cdots & 2p^{0+j} & \cdots \\ 0 & p^2 & 2p^3 & \cdots & 2p^{1+j} & \cdots \\ \vdots & & \ddots & & & \\ 0 & 0 & p^{2i} & \cdots & 2p^{i+j} & \cdots \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 & p^{2n} \end{pmatrix}$$

$$A + B = a\,2b^\top, \; a^\top = b^\top = (p^0, p^1, \ldots, p^{n-1})$$

Any set of columns (same set for rows) as NE support

Example $n = 3$, $p = 3$

fully mixed NE

1	6	18
0	9	54
0	0	81

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fully mixed NE

		1		
1	6	18		
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0	0	81		81

Any set of columns (same set for rows) as NE support

Example $n = 3$, $p = 3$

fully mixed NE

		3	**1**	
	1	**6**	**18**	
	0	**9**	**54**	**81**
	0	**0**	**81**	**81**

Any set of columns (same set for rows) as NE support

Example $n = 3$, $p = 3$

fully mixed NE

45	**3**	**1**	
1	**6**	**18**	**81**
0	**9**	**54**	**81**
0	**0**	**81**	**81**

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45	**3**	**1**	**/49**
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0	**0**	**81**	**81/49**

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45	3	1	$/49$
1	6	18	$81/49$
0	9	54	$81/49$
0	0	81	$81/49$

support = last row/column

		1	
1	6	18	
0	9	54	
0	0	81	81

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	0	1	
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fully mixed NE

45	3	1	/49
1	6	18	81/49
0	9	54	81/49
0	0	81	81/49

support = first & last row/column

63	0	1	
1	6	18	81
0	9	54	54
0	0	81	81

Any set of columns (same set for rows) as NE support

Example $n = 3$, $p = 3$

fully mixed NE

45	3	1	/49
1	6	18	81/49
0	9	54	81/49
0	0	81	81/49

support = first & last row/column

63	0	1	/64
1	6	18	81/64
0	9	54	54/64
0	0	81	81/64

Output efficiency

Given: game of rank **1**. It may have exponentially many equilibria.

Can all NE be computed in polynomial time in the size of input and output?

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Conjecture

Deciding if a rank-**1** game has more than one NE is NP-complete.

PPAD- versus NP-hardness

PPAD-hardness of NE seems **independent** of co-NP-hardness of uniqueness

rank **0**: zero-sum games, convex equilibrium set

rank **1**: one NE in polynomial time, exponentially many NE, conjecture: uniqueness co-NP-hard

rank $\geq$ **2**: PPAD-hard (proved for $\geq$ **3**)

rank $\boldsymbol{k}$: ε-approximate NE in time polynomial in input size and $\boldsymbol{k^{2k}}/\sqrt{\varepsilon}$

Further open problems

- Rank-**1** games can be solved **really fast.**
- Can they be used in applications of **really large** games?
- Economic interpretation of rank **1** payoffs?

E.g. discretized Cournot games:
players choose quantities $\boldsymbol{x}$, $\boldsymbol{y}$, get payoffs

$$\boldsymbol{x} \cdot (\mathbf{1} - \boldsymbol{x} - \boldsymbol{y}), \qquad \boldsymbol{y} \cdot (\mathbf{1} - \boldsymbol{x} - \boldsymbol{y})$$

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  E.g. discretized Cournot games:
  players choose quantities x, y, get payoffs

$$x \cdot (1 - x - y), \qquad y \cdot (1 - x - y),$$

  same best responses as rank-1 game

$$x - x^2 - xy \boxed{-y + y^2}, \qquad y - xy - y^2 \boxed{-x + x^2}$$

  (but no algorithm needed here).

Suggested example of a rank-**1** game

[References / comments very welcome]

row player 1 = seller with quality levels $i = 1, \ldots, m$

column player 2 = buyer with quantities bought $j = 1, \ldots, n$

price p_{ij} (paid by buyer to seller), quality level a_i, quantity b_j

payoff to player 1: $p_{ij} - \alpha a_i b_j \ (+ \gamma_j)$

payoff to player 2: $-p_{ij} + \beta a_i b_j \ (+ \delta_i)$

γ_j and δ_i irrelevant for best responses and Nash equilibria, omit.

$\Rightarrow$ **payoff sum** $(-\alpha + \beta) a_i b_j$ has rank **1**.

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$\Rightarrow$ **payoff sum** $(-\alpha + \beta) a_i b_j$ has rank **1**.

- But: how to interpret mixed Nash equilibria?

The Kohlberg-Mertens structure theorem (2 players)

Theorem The NE correspondence for $m \times n$ games is homeomorphic to the space of $m \times n$ games.

$$\big((A, B), \text{NE } (x, y)\big)$$

$$\updownarrow$$

$$(C, D)$$

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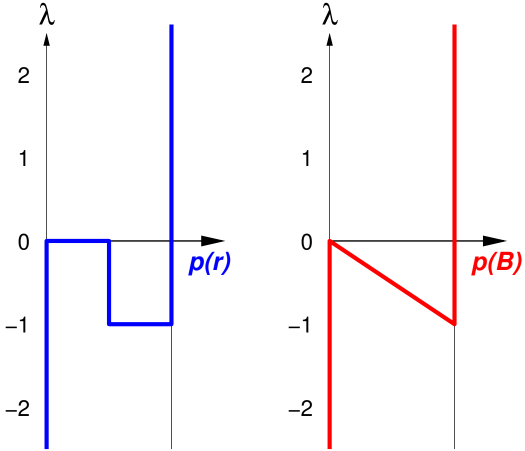
$$\updownarrow$$

$$(C, D)$$

A "slice" of the NE correspondence:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0+\lambda \\ 0 & 1+\lambda \end{pmatrix}$$



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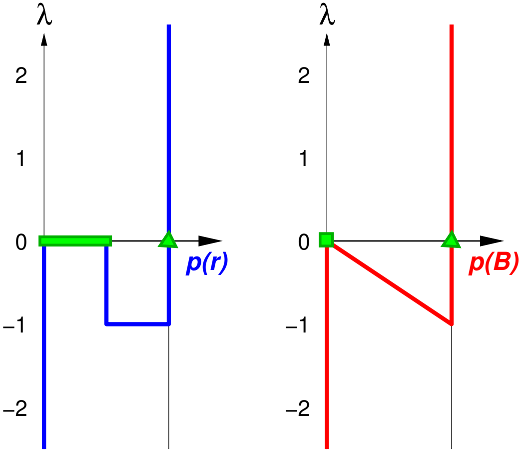
$$(C, D)$$

A “slice” of the NE correspondence:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 + \lambda \\ 0 & 1 + \lambda \end{pmatrix}$$



Main trick: parameterize game with $\boldsymbol{a} \in \mathbb{R}^{\boldsymbol{m}}$, $\boldsymbol{b} \in \mathbb{R}^{\boldsymbol{n}}$

$\boldsymbol{A} = \boldsymbol{a}\,\mathbf{1}^\top + \tilde{\boldsymbol{A}} \qquad (\tilde{\boldsymbol{A}}\mathbf{1} = \mathbf{0})$

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$$A = a\,\mathbf{1}^\top + \tilde{A} \qquad (\tilde{A}\mathbf{1} = \mathbf{0})$$

$$A = \begin{pmatrix} 0 & 8 \\ 4 & 6 \\ 6 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 \\ 5 & 5 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} -4 & 4 \\ -1 & 1 \\ 3 & -3 \end{pmatrix}$$

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$$B = \mathbf{1}b^\top + \tilde{B} \qquad (\mathbf{1}^\top \tilde{B} = \mathbf{0}^\top)$$

$$B = \begin{pmatrix} 4 & 0 \\ -1 & 0 \\ -15 & -18 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -6 \\ -4 & -6 \\ -4 & -6 \end{pmatrix} + \begin{pmatrix} 8 & 6 \\ 3 & 6 \\ -11 & -12 \end{pmatrix}$$

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(A, B), NE (x, y) $\leftrightarrow$ (C, D)

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$$(A, B), \text{ NE } (x, y) \;\leftrightarrow\; (C, D)$$

$$C = (Ay + x)\mathbf{1}^\top + \tilde{A}$$

$$D = \mathbf{1}(x^\top B + y^\top) + \tilde{B}$$

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$$A = a\mathbf{1}^\top + \tilde{A} \qquad (\tilde{A}\mathbf{1} = \mathbf{0})$$

$$A = \begin{pmatrix} 0 & 8 \\ 4 & 6 \\ 6 & 0 \end{pmatrix} \qquad y^\top = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

$$B = \mathbf{1}b^\top + \tilde{B} \qquad (\mathbf{1}^\top \tilde{B} = \mathbf{0}^\top)$$

$$B = \begin{pmatrix} 4 & 0 \\ -1 & 0 \\ -15 & -18 \end{pmatrix} \quad x = \begin{pmatrix} 0 \\ 3/4 \\ 1/4 \end{pmatrix}$$

(A, B), NE $(x, y) \leftrightarrow (C, D)$

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(A, B), NE $(x, y) \;\leftrightarrow\; (C, D)$

$C = (Ay + x)\mathbf{1}^\top + \tilde{A}$

$Ay + x =$

$$\begin{pmatrix} 2 \\ 4.5 \\ 4.5 \end{pmatrix} + \begin{pmatrix} 0 \\ 3/4 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5.25 \\ 4.75 \end{pmatrix}$$

$D = \mathbf{1}(x^\top B + y^\top) + \tilde{B}$

$x^\top B + y^\top$

$= \left(-\tfrac{18}{4}, -\tfrac{18}{4}\right) + \left(\tfrac{3}{4}, \tfrac{1}{4}\right)$

$= \left(-\tfrac{15}{4}, -\tfrac{17}{4}\right)$

Main trick: parameterize game with $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$

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$$B = \mathbf{1}\mathbf{b}^\top + \tilde{B} \qquad (\mathbf{1}^\top\tilde{B} = \mathbf{0}^\top)$$

$$B = \begin{pmatrix} 4 & 0 \\ -1 & 0 \\ -15 & -18 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} 0 \\ 3/4 \\ 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -6 \\ -4 & -6 \\ -4 & -6 \end{pmatrix} + \begin{pmatrix} 8 & 6 \\ 3 & 6 \\ -11 & -12 \end{pmatrix}$$

$$(A, B), \text{NE } (\mathbf{x}, \mathbf{y}) \;\leftrightarrow\; (C, D)$$

$$C = (A\mathbf{y} + \mathbf{x})\mathbf{1}^\top + \tilde{A}$$

$$A\mathbf{y} + \mathbf{x} =$$

$$\begin{pmatrix} 2 \\ 4.5 \\ 4.5 \end{pmatrix} + \begin{pmatrix} 0 \\ 3/4 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5.25 \\ 4.75 \end{pmatrix}$$

$$D = \mathbf{1}(\mathbf{x}^\top B + \mathbf{y}^\top) + \tilde{B}$$

$$\mathbf{x}^\top B + \mathbf{y}^\top$$

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The KM homeomorphism does not preserve $A + B$

$$A = \begin{pmatrix} 0 & 8 \\ 4 & 6 \\ 6 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 0 \\ -1 & 0 \\ -15 & -18 \end{pmatrix}, \qquad A + B = \begin{pmatrix} 4 & 8 \\ 3 & 6 \\ -9 & -18 \end{pmatrix},$$

$$A + B \neq C + D\,.$$

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$$A + B \neq C + D .$$

Theorem

There is a homeomorphism

(A, B), NE $(x, y) \iff (C, D)$

so that always $A + B = C + D$.

Parameterize **only** $\boldsymbol{A}$ with $\boldsymbol{a} \in \mathbb{R}^{\boldsymbol{m}}$ and $\boldsymbol{b} \in \mathbb{R}^{\boldsymbol{n}}$

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$$\boldsymbol{A} = \mathbf{1}\gamma\mathbf{1}^\top + \boldsymbol{a}\,\mathbf{1}^\top + \mathbf{1}\boldsymbol{b}^\top + \hat{\boldsymbol{A}}$$

$$\left(\mathbf{1}^\top \boldsymbol{a} = \mathbf{0}, \quad \boldsymbol{b}^\top \mathbf{1} = \mathbf{0}, \quad \hat{\boldsymbol{A}}\mathbf{1} = \mathbf{0}, \quad \mathbf{1}^\top \hat{\boldsymbol{A}} = \mathbf{0}^\top \right)$$

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$$\boldsymbol{A} = \begin{pmatrix} \mathbf{0} & \mathbf{8} \\ \mathbf{4} & \mathbf{6} \\ \mathbf{6} & \mathbf{0} \end{pmatrix}$$

Parameterize **only** $\boldsymbol{A}$ with $\boldsymbol{a} \in \mathbb{R}^{\boldsymbol{m}}$ and $\boldsymbol{b} \in \mathbb{R}^{\boldsymbol{n}}$

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$$\boldsymbol{A} = \begin{pmatrix} 0 & 8 \\ 4 & 6 \\ 6 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} -4 & 4 \\ 0 & 2 \\ 2 & -4 \end{pmatrix}$$

Parameterize **only** $\boldsymbol{A}$ with $\boldsymbol{a} \in \mathbb{R}^m$ and $\boldsymbol{b} \in \mathbb{R}^n$

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$$= \begin{pmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} -4 & 4 \\ -1 & 1 \\ 3 & -3 \end{pmatrix}$$

Parameterize **only** $\boldsymbol{A}$ with $\boldsymbol{a} \in \mathbb{R}^{m}$ and $\boldsymbol{b} \in \mathbb{R}^{n}$

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$$= \begin{pmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} -2/3 & 2/3 \\ -2/3 & 2/3 \\ -2/3 & 2/3 \end{pmatrix} + \begin{pmatrix} -10/3 & 10/3 \\ -1/3 & 1/3 \\ 11/3 & -11/3 \end{pmatrix}$$

Parameterize **only** A with $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$

$$A = \mathbf{1}\gamma\mathbf{1}^\top + a\mathbf{1}^\top + \mathbf{1}b^\top + \hat{A}$$

$$\left(\mathbf{1}^\top a = 0, \quad b^\top \mathbf{1} = 0, \quad \hat{A}\mathbf{1} = 0, \quad \mathbf{1}^\top \hat{A} = \mathbf{0}^\top \right)$$

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new homeomorphism $\boxed{(A, B), \text{ NE } (x, y) \Leftrightarrow (C, D)}$

$$C = \mathbf{1}\gamma\mathbf{1}^\top + \rho(Ay + x)\mathbf{1}^\top + \mathbf{1}\sigma(x^\top B + y^\top) + \hat{A}$$

$$D = (A + B) - C$$

Thank you!